

# Strichartz estimates for Schrödinger equations with variable coefficients and potentials at most linear at spatial infinity

Haruya Mizutani\*

## Abstract

In the present paper we consider Schrödinger equations with variable coefficients and potentials, where the principal part is a long-range perturbation of the flat Laplacian and potentials have at most linear growth at spatial infinity. We then prove local-in-time Strichartz estimates, outside a large compact set centered at origin, except for the endpoint. Moreover we also prove global-in-space Strichartz estimates under the non-trapping condition on the Hamilton flow generated by the kinetic energy.

## 1 Introduction

In this paper we study the so called (local-in-time) *Strichartz estimates* for the solutions to  $d$ -dimensional time-dependent Schrödinger equations

$$i\partial_t u(t) = Hu(t), \quad t \in \mathbb{R}; \quad u|_{t=0} = u_0 \in L^2(\mathbb{R}^d), \quad (1.1)$$

where  $d \geq 1$  and  $H$  is a Schrödinger operator with variable coefficients:

$$H = -\frac{1}{2} \sum_{j,k=1}^d \partial_{x_j} a^{jk}(x) \partial_{x_k} + V(x).$$

Throughout the paper we assume that  $a^{jk}(x)$  and  $V(x)$  are real-valued and smooth on  $\mathbb{R}^d$ , and  $(a^{jk}(x))$  is a symmetric matrix satisfying

$$C^{-1} \text{Id} \leq (a^{jk}(x)) \leq C \text{Id}, \quad x \in \mathbb{R}^d,$$

with some  $C > 0$ . We also assume

**Assumption A.** There exist constants  $\mu, \nu \geq 0$  such that, for any  $\alpha \in \mathbb{Z}_+^d$ ,

$$|\partial_x^\alpha (a^{jk}(x) - \delta_{jk})| \leq C_\alpha \langle x \rangle^{-\mu-|\alpha|}, \quad |\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{2-\nu-|\alpha|}, \quad x \in \mathbb{R}^d,$$

with some  $C_\alpha > 0$ .

We may assume  $\mu < 1$  and  $\nu < 2$  without loss of generality. It is well known that  $H$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^d)$  under Assumption A, and we denote the unique self-adjoint extension on  $L^2(\mathbb{R}^d)$  by the same symbol  $H$ . By the Stone theorem, the solution to (1.1) is given

---

2010 *Mathematics Subject Classification.* Primary 35Q41; Secondary 35B45, 81Q20.

*Key words and phrases.* Strichartz estimates, Schrödinger equation, unbounded potential.

\*Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan. E-mail: hmizutan@kurims.kyoto-u.ac.jp. Partly supported by GCOE ‘Fostering top leaders in mathematics’, Kyoto University.

by  $u(t) = e^{-itH}u_0$ , where  $e^{-itH}$  is a unique unitary group on  $L^2(\mathbb{R}^d)$  generated by  $H$  and called the propagator.

Let us recall the (global-in-time) Strichartz estimates for the free Schrödinger equation state that

$$\|e^{it\Delta/2}u_0\|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}, \quad (1.2)$$

where  $(p, q)$  satisfies the following *admissible* condition

$$2 \leq p, q \leq \infty, \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad (d, p, q) \neq (2, 2, \infty). \quad (1.3)$$

For  $d \geq 3$ ,  $(p, q) = (2, \frac{2d}{d-2})$  is called the endpoint. It is well known that these estimates are fundamental in studying the local well-posedness of Cauchy problem of nonlinear Schrödinger equations (see, *e.g.*, [6]). The estimates (1.2) were first proved by Strichartz [23] for a restricted pair of  $(p, q)$  with  $p = q = 2(d+2)/d$ , and have been extensively generalized for  $(p, q)$  satisfying (1.3) by [12, 15]. Moreover, in the flat case ( $a^{jk} \equiv \delta_{jk}$ ), local-in-time Strichartz estimates

$$\|e^{itH}u_0\|_{L^p([-T, T]; L^q(\mathbb{R}^d))} \leq C_T\|u_0\|_{L^2(\mathbb{R}^d)}, \quad (1.4)$$

have been extended to the case with potentials decaying at infinity [25] or increasing at most quadratically at infinity [26]. In particular, if  $V(x)$  has at most quadratic growth at spatial infinity, *i.e.*,

$$V \in C^\infty(\mathbb{R}^d; \mathbb{R}), \quad |\partial_x^\alpha V(x)| \leq C_\alpha \text{ for } |\alpha| \geq 2,$$

then it was shown by Fujiwara [11] that the fundamental solution  $E(t, x, y)$  of the propagator  $e^{-itH}$  satisfies

$$|E(t, x, y)| \lesssim |t|^{-d/2}, \quad x, y \in \mathbb{R}^d,$$

for  $t \neq 0$  small enough. The estimates (1.4) are immediate consequences of this estimate and the  $TT^*$ -argument due to Ginibre-Velo [12] (see Keel-Tao [15] for the endpoint estimate). For the case with magnetic fields or singular potentials, we refer to Yajima [26, 27] and references therein.

On the other hand, local-in-time Strichartz estimates on manifolds have recently been proved by many authors under several conditions on the geometry. Staffilani-Tataru [22], Robbiano-Zuily [18] and Bouclet-Tzvetkov [2] studied the case on the Euclidean space with the asymptotically flat metric under several settings. In particular, Bouclet-Tzvetkov [2] proved local-in-time Strichartz estimates without loss of derivatives under Assumption A with  $\mu > 0$  and  $\nu > 2$  and the non-trapping condition. Burq-Gérard-Tzvetkov [4] proved Strichartz estimates with a loss of derivative  $1/p$  on any compact manifolds without boundaries. They also proved that the loss  $1/p$  is optimal in the case of  $M = \mathbb{S}^d$ . Hassell-Tao-Wunsch [13] and the author [17] considered the case of non-trapping asymptotically conic manifolds which are non-compact Riemannian manifolds with an asymptotically conic structure at infinity. Bouclet [1] studied the case of an asymptotically hyperbolic manifold. Burq-Guillarmou-Hassell [5] recently studied the case of asymptotically conic manifolds with hyperbolic trapped trajectories of sufficiently small fractal dimension. For global-in-time Strichartz estimates, we refer to [10, 8] and the references therein in the case with electromagnetic potentials, and to [3, 24, 16] in the case of Euclidean space with an asymptotically flat metric.

The main purpose of the paper is to handle a mixed case of above two situations. More precisely, we show that local-in-time Strichartz estimates for long-range perturbations still hold (without loss of derivatives) if we add unbounded potentials which have at most linear growth at spatial infinity (*i.e.*,  $\nu \geq 1$ ), at least excluding the endpoint  $(p, q) = (2, 2d/(d-2))$ . To the best knowledge of the author, our result may be a first example on the case where both of variable coefficients and unbounded potentials in the spatial variable  $x$  are present.

To state the result, we recall the non-trapping condition. We denote by

$$H_0 = H - V = -\frac{1}{2} \sum_{j,k=1}^d \partial_{x_j} a^{jk}(x) \partial_{x_k}, \quad k(x, \xi) = \frac{1}{2} \sum_{j,k=1}^d a^{jk}(x) \xi_j \xi_k,$$

the principal part of  $H$  and the kinetic energy, respectively, and also denote by

$$(y_0(t, x, \xi), \eta_0(t, x, \xi))$$

the Hamilton flow generated by  $k(x, \xi)$ :

$$\dot{y}_0(t) = \partial_\xi k(y_0(t), \eta_0(t)), \quad \dot{\eta}_0(t) = -\partial_x k(y_0(t), \eta_0(t)); \quad (y_0(0), \eta_0(0)) = (x, \xi).$$

Note that the Hamiltonian vector field  $H_k$ , generated by  $k$ , is complete on  $\mathbb{R}^{2d}$  since  $(a^{jk})$  satisfies the uniform elliptic condition, and  $(y_0(t, x, \xi), \eta_0(t, x, \xi))$  hence exists for all  $t \in \mathbb{R}$ . We consider the following *non-trapping condition*:

$$\text{For any } (x, \xi) \in T^*\mathbb{R}^d \text{ with } \xi \neq 0, \quad |y_0(t, x, \xi)| \rightarrow +\infty \text{ as } t \rightarrow \pm\infty. \quad (1.5)$$

We now state our main result.

**Theorem 1.1.** (i) Suppose that  $H$  satisfies Assumption A with  $\mu > 0$  and  $\nu \geq 1$ . Then, there exist  $R_0 > 0$  large enough and  $\chi_0 \in C_0^\infty(\mathbb{R}^d)$  with  $\chi_0(x) = 1$  for  $|x| < R_0$  such that, for any  $T > 0$  and  $(p, q)$  satisfying (1.3) and  $p \neq 2$ , there exists  $C_T > 0$  such that

$$\|(1 - \chi_0)e^{-itH}u_0\|_{L^p([-T, T]; L^q(\mathbb{R}^d))} \leq C_T \|u_0\|_{L^2(\mathbb{R}^d)}. \quad (1.6)$$

(ii) Suppose that  $H$  satisfies Assumption A with  $\mu = \nu = 0$  and  $k(x, \xi)$  satisfies the non-trapping condition (1.5). Then, for any  $\chi \in C_0^\infty(\mathbb{R}^d)$ ,  $T > 0$  and  $(p, q)$  satisfying (1.3) and  $p \neq 2$ , we have

$$\|\chi e^{-itH}u_0\|_{L^p([-T, T]; L^q(\mathbb{R}^d))} \leq C_T \|u_0\|_{L^2(\mathbb{R}^d)}. \quad (1.7)$$

Moreover, combining with (1.6), we obtain global-in-space estimates

$$\|e^{-itH}u_0\|_{L^p([-T, T]; L^q(\mathbb{R}^d))} \leq C_T \|u_0\|_{L^2(\mathbb{R}^d)},$$

provided that  $\mu > 0$  and  $\nu \geq 1$ .

We here display the outline of the paper and explain the idea of the proof of Theorem 1.1. By the virtue of the Littlewood-Paley theory in terms of  $H_0$ , the proof of (1.6) can be reduced to that of following semi-classical Strichartz estimates:

$$\|(1 - \chi_0)\psi(h^2 H_0)e^{-itH}u_0\|_{L^p([-T, T]; L^q(\mathbb{R}^d))} \leq C_T \|u_0\|_{L^2(\mathbb{R}^d)}, \quad 0 < h \ll 1,$$

where  $\psi \in C_0^\infty(\mathbb{R})$  with  $\text{supp } \psi \Subset (0, \infty)$  and  $C_T > 0$  is independent of  $h$ . Moreover, there exists a smooth function  $a \in C^\infty(\mathbb{R}^{2d})$  supported in a neighborhood of the support of  $(1 - \chi_0)\psi \circ k$  such that  $(1 - \chi_0)\varphi(h^2 H_0)$  can be replaced with semi-classical pseudodifferential operator  $a(x, hD)$ . In Section 2, we collect some known results on the semi-classical pseudo-differential calculus and prove such a reduction to semi-classical estimates. Rescaling  $t \mapsto th$ , we want to show dispersive estimates for  $e^{itH}$  on a time scale of order  $h^{-1}$  for proving semi-classical Strichartz estimates. To prove dispersive estimates, we construct two kinds of parametrices, namely the Isozaki-Kitada and the WKB parametrices. Let  $a^\pm \in S(1, dx^2/\langle x \rangle^2 + d\xi^2/\langle \xi \rangle^2)$  be symbols supported in the following outgoing and incoming regions:

$$\{(x, \xi); |x| > R_0, \quad |\xi|^2 \in J, \quad \pm x \cdot \xi > -(1/2)|x||\xi|\},$$

respectively, where  $J \Subset (0, \infty)$  is an open interval so that  $\pi_\xi(\text{supp } \psi \circ k) \Subset J$  and  $\pi_\xi$  is the projection onto the  $\xi$ -space. If  $H$  is a long-range perturbation of  $-(1/2)\Delta$ , then the outgoing (resp. incoming) Isozaki-Kitada parametrix of  $e^{-itH}a^+(x, hD)$  for  $0 \leq t \leq h^{-1}$  (resp.  $e^{-itH}a^-(x, hD)$  for  $-h^{-1} \leq t \leq 0$ ) has been constructed by Robert [20] (see, also [2]). However, because of the unboundedness of  $V$  with respect to  $x$ , it is difficult to construct such parametrices of  $e^{-itH}a^\pm(x, hD)$ . To overcome this difficulty, we use a method due to Yajima-Zhang [29] as follows. We approximate  $e^{-itH}$  by  $e^{-itH_h}$ , where  $H_h = H - V + V_h$  and  $V_h$  vanishes in the region  $\{x; |x| \gg h^{-1}\}$ . Suppose that  $a^+$  (resp.  $a^-$ ) is supported in the intersection of the outgoing (resp. incoming) region and  $\{x; |x| < h^{-1}\}$ . In Section 3, we construct the Isozaki-Kitada parametrix of  $e^{-itH_h}a^\pm(x, hD)$  for  $0 \leq \pm t \leq h^{-1}$  and prove the following justification of the approximation: for any  $N > 0$ ,

$$\sup_{0 \leq \pm t \leq h^{-1}} \|(e^{-itH} - e^{-itH_h})a^\pm(x, hD)f\|_{L^2} \leq C_N h^N \|f\|_{L^2}, \quad 0 < h \ll 1.$$

In Section 4, we discuss the WKB parametrix construction of  $e^{-itH}a(x, hD)$  on a time scale of order  $h^{-1}$ , where  $a$  is supported in  $\{(x, \xi); |x| > h^{-1}, |\xi|^2 \in I\}$ . Such a parametrix construction is basically known for the potential perturbation case (see, *e.g.*, [28]) and has been proved by the author for the case on asymptotically conic manifolds [17]. Combining these results studied in Sections 2, 3 and 4 with the Keel-Tao theorem [15], we prove semi-classical Strichartz estimates in Section 5. Section 6 is devoted to the proof of (1.7). The proof heavily depends on local smoothing effects due to Doi [9] and the Chirist-Kiselev lemma [7]. The method of the proof is similar as that in Robbiano-Zuily [18]. Appendix A is devoted to prove some technical inequalities on the Hamilton flow needed for constructing the WKB parametrix.

Throughout the paper we use the following notations. For  $A, B \geq 0$ ,  $A \lesssim B$  means that there exists some universal constant  $C > 0$  such that  $A \leq CB$ . We denote the set of multi-indices by  $\mathbb{Z}_+^d$ . For Banach spaces  $X$  and  $Y$ ,  $\mathcal{L}(X, Y)$  denotes the Banach space of bounded operators from  $X$  to  $Y$ , and we write  $\mathcal{L}(X) := \mathcal{L}(X, X)$ .

## 2 Reduction to semi-classical estimates

We here show that Theorem 1.1 (i) follows from semi-classical Strichartz estimates. We first record known results on the pseudo-differential calculus and the  $L^p$ -functional calculus. For any symbol  $a \in C^\infty(\mathbb{R}^{2d})$  and  $h \in (0, 1]$ , we denote the semi-classical pseudo-differential operator ( $h$ -PDO for short) by  $a(x, hD_x)$ :

$$a(x, hD_x)u(x) = (2\pi h)^{-d} \int e^{i(x-y) \cdot \xi / h} a(x, \xi) u(y) dy d\xi, \quad u \in \mathcal{S}(\mathbb{R}^d),$$

where  $\mathcal{S}(\mathbb{R}^d)$  is the Schwartz class. For a metric

$$g = dx^2 / \langle x \rangle^2 + d\xi^2 / \langle \xi \rangle^2 \text{ on } T^*\mathbb{R}^d,$$

we consider Hörmander's symbol class  $S(m, g)$  with a weighted function  $m$ , namely we write  $a \in S(m, g)$  if  $a \in C^\infty(\mathbb{R}^{2d})$  and

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta} m(x, \xi) \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|}, \quad x, \xi \in \mathbb{R}^d.$$

Let  $a \in S(m_1, g)$ ,  $b \in S(m_2, g)$ . For any  $N = 0, 1, 2, \dots$ , the symbol of the composition  $a(x, hD)b(x, hD)$ , denoted by  $a \sharp b$ , has an asymptotic expansion

$$a \sharp b(x, \xi) = \sum_{|\alpha| \leq N} \frac{h^{|\alpha|}}{i^{|\alpha|} \alpha!} \partial_\xi^\alpha a(x, \xi) \cdot \partial_x^\alpha b(x, \xi) + h^{N+1} r_N(x, \xi) \quad (2.1)$$

with some  $r_N \in S(\langle x \rangle^{-N-1} \langle \xi \rangle^{-N-1} m_1 m_2, g)$ . For  $a \in S(1, g)$ ,  $a(x, hD_x)$  is extended to a bounded operator on  $L^2(\mathbb{R}^d)$ . Moreover, if  $a \in S(\langle \xi \rangle^{-N}, g)$  for some  $N > d$ , then the distribution kernel  $A_h(x, y)$  of  $a(x, hD)$  satisfies

$$\sup_x \int |A_h(x, y)| dy + \sup_y \int |A_h(x, y)| dx \leq C$$

for some  $C > 0$  independent of  $h$ . By using this estimate, the Schur lemma and an interpolation, we have

$$\|a(x, hD)\|_{\mathcal{L}(L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))} \leq C_{qr} h^{-d(1/q-1/r)}, \quad 1 \leq q \leq r \leq \infty, \quad h \in (0, 1],$$

where  $C_{qr} > 0$  is independent of  $h$ .

We next consider the  $L^p$ -functional calculus. The following lemma, which has been proved by [2, Proposition 2.5], tells us that for any  $\varphi \in C_0^\infty(\mathbb{R})$  with  $\text{supp } \varphi \Subset (0, \infty)$ ,  $\varphi(h^2 H_0)$  can be approximated in terms of the  $h$ -PDO.

**Lemma 2.1.** *Let  $\varphi \in C_0^\infty(\mathbb{R})$ ,  $\text{supp } \varphi \Subset (0, \infty)$  and  $N \geq 0$  a non-negative integer. Then there exist symbols  $a_j \in S(1, g)$ ,  $j = 0, 1, \dots, N$ , such that*

- (i)  $a_0(x, \xi) = \varphi(k(x, \xi))$  and  $a_j(x, \xi)$  are supported in the support of  $\varphi(k(x, \xi))$  for any  $j$ .
- (ii) For every  $1 \leq q \leq r \leq \infty$  there exists  $C_{qr} > 0$  such that

$$\|a_j(x, hD_x)\|_{\mathcal{L}(L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))} \leq C_{qr} h^{-d(1/q-1/r)},$$

uniformly with respect to  $h \in (0, 1]$ .

- (iii) There exists a constant  $N_0 \geq 0$  such that, for all  $1 \leq q \leq r \leq \infty$ ,

$$\|\varphi(h^2 H_0) - a(x, hD_x)\|_{\mathcal{L}(L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))} \leq C_{Nqr} h^{N-N_0-d(1/q-1/r)}$$

uniformly with respect to  $h \in (0, 1]$ , where  $a = \sum_{j=0}^N h^j a_j$ .

**Remark 2.2.** We note that Assumption A implies a stronger bounds on  $a_j$ :

$$|\partial_x^\alpha \partial_\xi^\beta a_j(x, \xi)| \leq C_{\alpha\beta} \langle x \rangle^{-j-|\alpha|} \langle \xi \rangle^{-|\beta|},$$

though we do not use this estimate in the following argument.

We next recall the Littlewood-Paley decomposition in terms of  $\varphi(h^2 H_0)$ . Consider a 4-adic partition of unity with respect to  $[1, \infty)$ :

$$\sum_{j=0}^{\infty} \varphi(2^{-2j} \lambda) = 1, \quad \lambda \in [1, \infty),$$

where  $\varphi \in C_0^\infty(\mathbb{R})$  with  $\text{supp } \varphi \subset [1/4, 4]$  and  $0 \leq \varphi \leq 1$ .

**Lemma 2.3.** *Let  $\chi \in C_0^\infty(\mathbb{R}^d)$ . Then, for all  $2 \leq q < \infty$  with  $0 \leq d(1/2 - 1/q) \leq 1$ ,*

$$\|(1 - \chi)f\|_{L^q(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)} + \left( \sum_{j=0}^{\infty} \|(1 - \chi)\varphi(2^{-2j} H_0)f\|_{L^q(\mathbb{R}^d)}^2 \right)^{\frac{1}{2}}.$$

This lemma can be proved similarly to the case of the Laplace-Beltrami operator on compact manifolds without boundaries (cf. [4, Corollary 2.3]). By using this lemma, we have the following:

**Proposition 2.4.** *Let  $\chi_0$  be as that in Theorem 1.1. Suppose that there exist  $h_0, \delta > 0$  small enough such that, for any  $\psi \in C^\infty((0, \infty))$  and any admissible pair  $(p, q)$  with  $p > 2$ ,*

$$\|(1 - \chi_0)\psi(h^2 H_0)e^{-itH}u_0\|_{L^p([- \delta, \delta]; L^q(\mathbb{R}^d))} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}, \quad (2.2)$$

uniformly with respect to  $h \in (0, h_0]$ . Then, the statement of Theorem 1.1 (i) holds.

*Proof.* By Lemma 2.3 with  $f = e^{-itH}u_0$ , the Minkowski inequality and the unitarity of  $e^{-itH}$  on  $L^2(\mathbb{R}^d)$ , we have

$$\begin{aligned} & \| (1 - \chi_0) e^{-itH} u_0 \|_{L^p([- \delta, \delta]; L^q(\mathbb{R}^d))} \\ & \lesssim \| u_0 \|_{L^2(\mathbb{R}^d)} + \left( \sum_{j=0}^{\infty} \| (1 - \chi_0) \varphi(2^{-2j} H_0) e^{-itH} u_0 \|_{L^p([- \delta, \delta]; L^q(\mathbb{R}^d))}^2 \right)^{1/2}. \end{aligned}$$

For  $0 \leq j \leq [-\log h_0] + 1$ , we have the bound

$$\begin{aligned} & \sum_{j=0}^{[-\log h_0]+1} \| (1 - \chi_0) \varphi(2^{-2j} H_0) e^{-itH} u_0 \|_{L^p([- \delta, \delta]; L^q(\mathbb{R}^d))}^2 \\ & \lesssim \sum_{j=0}^{[-\log h_0]+1} \| \varphi(2^{-2j} H_0) \|_{\mathcal{L}(L^2(\mathbb{R}^d), L^q(\mathbb{R}^d))} \| e^{-itH} u_0 \|_{L^\infty([- \delta, \delta]; L^2(\mathbb{R}^d))} \\ & \lesssim ([-\log h_0] + 1) 2^{([- \log h_0] + 1)d(1/2 - 1/q)} \| u_0 \|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Choosing  $\psi \in C_0^\infty(\mathbb{R})$  with  $\psi \equiv 1$  on  $\text{supp } \varphi$ , we can write

$$\begin{aligned} & \varphi(h^2 H_0) e^{-itH} \\ & = \psi(h^2 H_0) e^{-itH} \varphi(h^2 H_0) + \psi(h^2 H_0) i \int_0^t e^{-i(t-s)H} [V, \varphi(h^2 H_0)] e^{-isH} ds \\ & = \psi(h^2 H_0) e^{-itH} \varphi(h^2 H_0) + R(t, h). \end{aligned}$$

Since  $[H, \varphi(h^2 H_0)] = [V, \varphi(h^2 H_0)] = O(h)$  on  $L^2(\mathbb{R}^d)$ , the remainder term  $R(t, h)$  satisfies

$$\begin{aligned} & \sup_{0 \leq t \leq 1} \| R(t, h) \|_{\mathcal{L}(L^2(\mathbb{R}^d), L^q(\mathbb{R}^d))} \\ & \lesssim \| \psi(h^2 H_0) \|_{\mathcal{L}(L^2(\mathbb{R}^d), L^q(\mathbb{R}^d))} \| [V, \varphi(h^2 H_0)] \|_{\mathcal{L}(L^2(\mathbb{R}^d))} \\ & \lesssim h^{-d(1/2 - 1/q) + 1}. \end{aligned} \tag{2.3}$$

We here note that  $\gamma := -d(1/2 - 1/q) + 1 = -2/p + 1 > 0$  since  $p > 2$ . By (2.2), (2.3) with  $h = 2^{-j}$  and the almost orthogonality of  $\text{supp } \varphi(2^{-2j} \cdot)$ , we obtain

$$\begin{aligned} & \sum_{j=[-\log h_0]}^{\infty} \| (1 - \chi_0) \varphi(2^{-2j} H_0) e^{-itH} u_0 \|_{L^p([- \delta, \delta]; L^q(\mathbb{R}^d))}^2 \\ & \lesssim \sum_{j=[-\log h_0]}^{\infty} \left( \| \varphi(2^{-2j} H_0) u_0 \|_{L^2(\mathbb{R}^d)}^2 + 2^{-2\gamma j} \| u_0 \|_{L^2(\mathbb{R}^d)}^2 \right) \\ & \lesssim \| u_0 \|_{L^2(\mathbb{R}^d)}^2, \end{aligned}$$

Combining with the bound for  $0 \leq j \leq [-\log h_0] + 1$ , we have

$$\| (1 - \chi_0) e^{-itH} u_0 \|_{L^p([- \delta, \delta]; L^q(\mathbb{R}^d))} \lesssim \| u_0 \|_{L^2(\mathbb{R}^d)}.$$

Finally, we split the time interval  $[-T, T]$  into  $([T/\delta] + 1)$  intervals with size  $2\delta$ , and obtain

$$\begin{aligned} & \| (1 - \chi_0) \psi(h^2 H_0) e^{-itH} u_0 \|_{L^p([-T, T]; L^q(\mathbb{R}^d))} \\ & \leq \sum_{k=[T/\delta]}^{[T/\delta]+1} \| (1 - \chi_0) \psi(h^2 H_0) e^{-itH} e^{-i(k+1)H} u_0 \|_{L^p([- \delta, \delta]; L^q(\mathbb{R}^d))} \\ & \leq C_T \| u_0 \|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

□

### 3 Isozaki-Kitada parametrix

In this section we assume Assumption A with  $0 < \mu = \nu < 1/2$  without loss of generality, and construct the Isozaki-Kitada parametrix. Since the potential  $V$  can grow at infinity, it is difficult to construct directly the Isozaki-Kitada parametrix for  $e^{-itH}$  even though we restrict it in an outgoing or incoming region. To overcome this difficulty, we approximate  $e^{-itH}$  as follows. Let  $\rho \in C_0^\infty(\mathbb{R}^d)$  be a cut-off function such that  $\rho(x) = 1$  if  $|x| \leq 1$  and  $\rho(x) = 0$  if  $|x| \geq 2$ . For a small constant  $\varepsilon > 0$  and  $h \in (0, 1]$ , we define  $H_h$  by

$$H_h = H_0 + V_h, \quad V_h = V(x)\rho(\varepsilon hx).$$

We note that, for any fixed  $\varepsilon > 0$ ,

$$h^2 |\partial_x^\alpha V_h(x)| \leq C_\alpha h^2 \langle x \rangle^{2-\mu-|\alpha|} \leq C_{\varepsilon, \alpha} \langle x \rangle^{-\mu-|\alpha|}, \quad x \in \mathbb{R}^d,$$

where  $C_{\varepsilon, \alpha}$  may be taken uniformly with respect to  $h \in (0, 1]$ . Such a type modification has been used to prove Strichartz estimates and local smoothing effects for Schrödinger equations with super-quadratic potentials (see, Yajima-Zhang [29, Section 4]).

For  $R > 0$ , an open interval  $J \Subset (0, \infty)$  and  $-1 < \sigma < 1$ , we define the outgoing and incoming regions by

$$\Gamma^\pm(R, J, \sigma) := \left\{ (x, \xi) \in \mathbb{R}^{2d}; |x| > R, |\xi| \in J, \pm \frac{x \cdot \xi}{|x||\xi|} > -\sigma \right\},$$

respectively. Since  $H_0 + h^2 V_h$  is a long-range perturbation of  $-\Delta/2$ , we have the following theorem due to Robert [20] and Bouclet-Tzvetkov [2].

**Theorem 3.1.** *Let  $J, J_0, J_1$  and  $J_2$  be relatively compact open intervals,  $\sigma, \sigma_0, \sigma_1$  and  $\sigma_2$  real numbers so that  $J \Subset J_0 \Subset J_1 \Subset J_2 \Subset (0, \infty)$  and  $-1 < \sigma < \sigma_0 < \sigma_1 < \sigma_2 < 1$ . Fix arbitrarily  $\varepsilon > 0$ . Then there exist  $R_0 > 0$  large enough and  $h_0 > 0$  small enough such that the followings hold.*

(i) *There exist two families of smooth functions*

$$\{S_h^+; h \in (0, h_0], R \geq R_0\}, \quad \{S_h^-; h \in (0, h_0], R \geq R_0\} \subset C^\infty(\mathbb{R}^{2d}; \mathbb{R})$$

*satisfying the Eikonal equation associated to  $k + h^2 V_h$ :*

$$k(x, \partial_x S_h^\pm(x, \xi)) + h^2 V_h(x) = \frac{1}{2} |\xi|^2, \quad (x, \xi) \in \Gamma^\pm(R^{1/4}, J_2, \sigma_2), \quad h \in (0, h_0],$$

*respectively, such that*

$$|\partial_x^\alpha \partial_\xi^\beta (S_h^\pm(x, \xi) - x \cdot \xi)| \leq C_{\alpha\beta} \langle x \rangle^{1-\mu-|\alpha|}, \quad \alpha, \beta \in \mathbb{Z}_+^d, \quad x, \xi \in \mathbb{R}^d, \quad (3.1)$$

*where  $C_{\alpha\beta} > 0$  may be taken uniformly with respect to  $R$  and  $h$ .*

(ii) *For every  $R \geq R_0$ ,  $h \in (0, h_0]$  and  $N = 0, 1, \dots$ , we can find*

$$b_h^\pm = \sum_{j=0}^N h^j b_{h,j}^\pm \in S(1, g) \quad \text{with} \quad \text{supp } b_{h,j}^\pm \subset \Gamma^\pm(R^{1/3}, J_1, \sigma_1)$$

*such that, for every  $a^\pm \in S(1, g)$  with  $\text{supp } a^\pm \subset \Gamma^\pm(R, J, \sigma)$ , there exist*

$$c_h^\pm = \sum_{j=0}^N h^j c_{h,j}^\pm(h) \in S(1, g) \quad \text{with} \quad \text{supp } c_{h,j}^\pm \subset \Gamma^\pm(R^{1/2}, J_0, \sigma_0)$$

such that, for all  $\pm t \geq 0$ ,

$$e^{-ithH_h} a^\pm(x, hD) = \mathcal{F}_{\text{IK}}(S_h^\pm, b_h^\pm) e^{ith\Delta/2} \mathcal{F}_{\text{IK}}(S_h^\pm, c_h^\pm)^* + Q_{\text{IK}}^\pm(t, h, N),$$

respectively, where  $\mathcal{F}_{\text{IK}}(S_h^\pm, w)$  are Fourier integral operators defined by

$$\mathcal{F}_{\text{IK}}(S_h^\pm, w)f(x) = \frac{1}{(2\pi h)^d} \int e^{i(S_h^\pm(x, \xi) - y \cdot \xi)/h} w(x, \xi) f(y) dy d\xi,$$

respectively. Moreover, for all  $s \in \mathbb{R}$  there exists  $C_N > 0$  such that

$$\|(h^2 H_h + L)^s Q_{\text{IK}}^\pm(t, h, N)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_N h^{N-1} \quad (3.2)$$

uniformly with respect to  $h \in (0, h_0]$  and  $0 \leq \pm t \leq h^{-1}$ , where  $L > 1$ , independent of  $h$ ,  $t$  and  $x$ , is a large constant so that  $h^2 V_h + L \geq 1$ .

(iii) The distribution kernels  $K_{\text{IK}}^\pm(t, h, x, y)$  of  $\mathcal{F}_{\text{IK}}(S_h^\pm, b_h^\pm) e^{-ith\Delta/2} \mathcal{F}_{\text{IK}}(S_h^\pm, c_h^\pm)^*$  satisfy dispersive estimates:

$$|K_{\text{IK}}^\pm(t, h, x, y)| \leq C |th|^{-d/2}, \quad (3.3)$$

for any  $h \in (0, h_0]$ ,  $0 \leq \pm t \leq h^{-1}$  and  $x, \xi \in \mathbb{R}^d$ , respectively.

*Proof.* This theorem is basically known, and we only check (3.2) for the outgoing case. For the detail of the proof, we refer to [20, Section 4] and [2, Section 3]. We also refer to the original paper by Isozaki-Kitada [14].

The remainder  $Q_{\text{IK}}^+(t, h, N)$  consists of the following three parts:

$$\begin{aligned} & - h^{N+1} e^{-ithH_h} q_1(h, x, hD), \\ & - ih^N \int_0^t e^{-i(t-\tau)hH_h} \mathcal{F}_{\text{IK}}^+(S_h^+, q_2(h)) e^{i\tau h\Delta/2} \mathcal{F}_{\text{IK}}^+(S_h^+, c_h^+)^* d\tau, \\ & - (i/h) \int_0^t e^{-i(t-\tau)hH_h} \tilde{Q}(\tau, h) d\tau, \end{aligned}$$

where  $\{q_1(h, \cdot, \cdot), q_2(h, \cdot, \cdot); h \in (0, h_0]\} \subset \bigcap_{M=1}^\infty S(\langle x \rangle^{-N} \langle \xi \rangle^{-M}, g)$  is a bounded set, and  $\tilde{Q}(s, h)$  is a integral operator with a kernel  $\tilde{q}(s, h, x, y)$  satisfying

$$|\partial_x^\alpha \partial_\xi^\beta \tilde{q}(\tau, h, x, y)| \leq C_{\alpha\beta} h^{M-|\alpha+\beta|} (1 + |\tau| + |x| + |y|)^{-M+|\alpha+\beta|}, \quad \tau \geq 0,$$

for any  $M \geq 0$ . A standard  $L^2$ -boundedness of  $h$ -PDO and FIO then imply

$$\|(h^2 H_0 + 1)^s (q_1(h, x, hD) + \mathcal{F}_{\text{IK}}^+(S_h^+, q_2(h)))\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_s,$$

and a direct computation yields

$$\|(h^2 H_0 + 1)^s \tilde{Q}(\tau, h)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_M h^M.$$

On the other hand, if we choose a constant  $L > 0$  so large that  $h^2 V_h + L \geq 1$ , then we have

$$\|(h^2 H_h + L)^s (h^2 H_0 + 1)^{-s}\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_s. \quad (3.4)$$

Indeed, if  $s$  is a positive integer, then (3.4) is obvious since  $h^2 V_h + L \lesssim 1$ . For any negative integer  $s$ , (3.4) follows from the fact that  $h^2 H_0 + 1 \leq h^2 H_h + L$ . For general  $s \in \mathbb{R}$ , we obtain (3.4) by an interpolation. (3.2) follows from the above three estimates since  $(h^2 H_h + L)^s$  commutes with  $e^{-ithH_h}$ .  $\square$



The following key lemma tells us that one can still construct the Isozaki-Kitada parametrrix of the original propagator  $e^{-ithH}$  if we restrict the support of initial data in the region  $\{x; |x| < h^{-1}\}$ .

**Lemma 3.2.** *Suppose that  $\{a_h^\pm\}_{h \in (0,1]}$  are bounded sets in  $S(1, g)$  and satisfy*

$$\text{supp } a_h^\pm \subset \Gamma^\pm(R, J, \sigma) \cap \{x; |x| < h^{-1}\},$$

*respectively. Then for any  $M \geq 0$ ,  $h \in (0, h_0]$  and  $0 \leq \pm t \leq h^{-1}$ , we have*

$$\|(e^{-ithH} - e^{-ithH_h})a_h^\pm(x, hD)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_M h^M,$$

*where  $C_M > 0$  is independent of  $h$  and  $t$ .*

*Proof.* We prove the lemma for the outgoing case only, and the proof of incoming case is completely analogous. We set  $A = a_h^+(x, hD)$  and  $W_h = V - V_h$ . The Duhamel formula yields

$$\begin{aligned} & (e^{-ithH} - e^{-ithH_h})A \\ &= -ih \int_0^t e^{-i(t-s)hH} W_h e^{-ishH_h} A ds \\ &= -ih \int_0^t e^{-i(t-s)hH} e^{-ishH_h} W_h A ds \\ &\quad - h^2 \int_0^t e^{-i(t-s)hH} \int_0^s e^{-i(s-\tau)hH_h} [H_0, W_h] e^{-i\tau hH_h} A d\tau ds. \end{aligned}$$

Since  $\text{supp } a_h^+(\cdot, \xi) \subset \{x; |x| < h^{-1}\}$ , we learn  $\text{supp } W_h \cap a_h^+(\cdot, \xi) = \emptyset$  if  $\varepsilon < 1$ . Combining with the asymptotic formula (2.1), this support property implies

$$\|W_h A\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_M h^M$$

for any  $M \geq 0$ . A direct computation yields that  $[H_0, W_h]$  is of the form

$$\sum_{|\alpha|=0,1} a_\alpha(x) \partial_x^\alpha, \quad \text{supp } a_\alpha \subset \text{supp } W_h, \quad |\partial_x^\beta a_\alpha(x)| \leq C_{\alpha\beta} \langle x \rangle^{-\mu+|\alpha|-|\beta|}.$$

The support property of  $W_h$  again yields

$$\|[H_h, [H_0, W_h]]A\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_M h^M.$$

We next consider  $[H_h, [K, W_h]]$  which has the form

$$\sum_{|\alpha|=1,2} b_\alpha(x) \partial_x^\alpha + W_1(x),$$

where  $b_\alpha$  and  $W_1$  are supported in  $\text{supp } W_h$  and satisfy

$$|\partial_x^\beta b_\alpha(x)| \leq C_{\alpha\beta} \langle x \rangle^{-2-\mu+|\alpha|-|\beta|}, \quad |\partial_x^\beta W_1(x)| \leq C_{\alpha\beta} \langle x \rangle^{2-2\mu}.$$

Setting  $I_1 = \sum_{|\alpha|=1,2} b_\alpha(x) \partial_x^\alpha$  and  $N_\mu := [1/\mu] + 1$ , we iterate this procedure  $N_\mu$  times with  $W_h$  replaced by  $W_1$ .  $(e^{-ithH} - e^{-ithH_h})A$  then can be brought to a linear combination of the following forms (modulo  $O(h^M)$  on  $L^2(\mathbb{R}^d)$ ):

$$\int_{t \geq s_1 \geq \dots \geq s_j \geq 0} e^{-i(t-s_1)hH} e^{-i(s_1-s_j)hH_h} I_{j/2} e^{-is_j hH_h} A ds_j \dots ds_1$$

for  $j = 2m$ ,  $m = 1, 2, \dots, N_\mu$ , and

$$\int_{t \geq s_1 \geq \dots \geq s_{N_\mu} \geq 0} e^{-i(t-s_1)hH} e^{-i(s_1-s_{N_\mu})hH_h} W_{N_\mu} e^{-is_{N_\mu} hH_h} A ds_{2N_\mu} \dots ds_1,$$

where  $I_k$  are second order differential operators with smooth and bounded coefficients, and  $W_{N_\mu}$  is a bounded function since  $2 - 2\mu N_\mu < 0$ . Moreover, they are supported in  $\{x; |x| > (\varepsilon h)^{-1}\}$ . Therefore, it is sufficient to show that, for any  $h \in (0, h_0]$ ,  $0 \leq \tau \leq h^{-1}$ ,  $\alpha \in \mathbb{Z}_+^d$  and  $M \geq 0$ ,

$$\|(1 - \rho(\varepsilon h x)) \partial_x^\alpha e^{-i\tau h H_h} A\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_{M,\alpha} h^{M-|\alpha|}. \quad (3.5)$$

We now apply Theorem 3.1 to  $e^{-i\tau h H_h} A$  and obtain

$$e^{-i\tau h H_h} A = \mathcal{F}_{\text{IK}}(S_h^+, b_h^+) e^{i\tau h \Delta/2} \mathcal{F}_{\text{IK}}(S_h^+, c_h^+)^* + Q_{\text{IK}}^+(t, h, N).$$

Recall that the elliptic nature of  $H_0$  implies, for every  $s \geq 0$ ,

$$\begin{aligned} \|\langle D \rangle^s (h^2 H_0 + 1)^{-s/2} f\|_{L^2(\mathbb{R}^d)} &\leq C h^{-s} \|f\|_{L^2(\mathbb{R}^d)}, \\ \|(h^2 H_0 + 1)^{s/2} (h^2 H_h + L)^{-s/2} f\|_{L^2(\mathbb{R}^d)} &\leq C \|f\|_{L^2(\mathbb{R}^d)}, \end{aligned}$$

if  $L > 0$  so large that  $h^2 H_h + L \geq 1$ . Combining these estimates with (3.2), the remainder satisfies

$$\|\langle D \rangle^s Q_{\text{IK}}^+(t, h, N) f\|_{L^2(\mathbb{R}^d)} \leq C_{N,s} h^{N-1-s} \|f\|_{L^2(\mathbb{R}^d)}, \quad s \geq 0.$$

The main term can be handled in terms of the non-stationary phase method as follows. The distribution kernel of the main term is given by

$$(2\pi h)^{-d} (1 - \rho(\varepsilon h x)) \partial_x^\alpha \int e^{i\Phi_h^+(\tau, x, y, \xi)/h} b_h^+(x, \xi) \overline{c_h^+(y, \xi)} d\xi, \quad (3.6)$$

where  $\Phi_h^+(\tau, x, y, \xi) = S_h^+(x, \xi) - \frac{1}{2}\tau|\xi|^2 - S_h^+(y, \xi)$ . We here claim that

$$\text{supp } c_h^+ \subset \{(x, \xi) \in \mathbb{R}^{2d}; a_h^+(x, \partial_\xi S_h^+(x, \xi)) \neq 0\}. \quad (3.7)$$

This property follows from the construction of  $c_j^+(h)$ ,  $j = 0, 1, \dots, N$ . We set

$$\tilde{S}_h^+(x, y, \xi) = \int_0^1 \partial_x S_h^+(y + \theta(x - y), \xi) d\theta.$$

Let  $\xi \mapsto [\tilde{S}_h^+]^{-1}(x, y, \xi)$  be the inverse map of  $\xi \mapsto \tilde{S}_h^+(x, y, \xi)$ , and we denote their Jacobians by  $A_1 = |\det \partial_\xi \tilde{S}_h^+(x, y, \xi)|$  and  $A_2 = |\det \partial_\xi [\tilde{S}_h^+]^{-1}(x, y, \xi)|$ , respectively.  $c_j^+(h)$  then satisfy the following triangular system:

$$\overline{c_{h,j}^+(x, \xi)} = b_{h,0}^+(x, \xi)^{-1} \left( r_{h,j}^+(x, \tilde{S}_h^+(x, y, \xi)) A_1 \right) \Big|_{y=x}, \quad j = 0, 1, \dots, N,$$

where  $r_{h,0}^+ = a_h^+(x, \tilde{S}_h^+(x, y, \xi))$  and  $r_j^+$ ,  $j \geq 1$ , is a linear combination of

$$\frac{1}{i^{|\alpha|} \alpha!} \left( \partial_\xi^\alpha \partial_y^\alpha b_{h,k_0}^+(x, [\tilde{S}_h^+]^{-1}(x, y, \xi)) c_{h,k_1}^+(y, [\tilde{S}_h^+]^{-1}(x, y, \xi)) A_2 \right) \Big|_{y=x},$$

where  $\alpha \in \mathbb{Z}_+^d$  and  $k_0, k_1 = 0, 1, \dots, j$  so that  $0 \leq |\alpha| \leq j$ ,  $k_0 + k_1 = j - |\alpha|$  and  $k_1 \leq j - 1$ . Therefore, we inductively obtain

$$\text{supp } c_{h,0}^+ \subset \text{supp } r_0^+|_{y=x}, \quad \text{supp } c_{h,j}^+ \subset \text{supp } c_{h,j-1}^+(h), \quad j = 1, 2, \dots, N,$$

and (3.7) follows. In particular,  $c_h^+$  vanishes in the region  $\{x; |x| \geq h^{-1}\}$ . By using (3.1), we have

$$\partial_\xi \Phi_h^+(\tau, x, y, \xi) = (x - y)(\text{Id} + O(R^{-\mu/3})) - \tau \xi,$$

which implies

$$|\partial_\xi \Phi_h^+(\tau, x, y, \xi)| \geq \frac{|x|}{2} - |y| - |\tau\xi|$$

as long as  $R \geq 1$  large enough. We now set

$$\varepsilon = \frac{1}{2(\sup J_2)^{1/2} + 2}.$$

Since  $|x| > (\varepsilon h)^{-1}$ ,  $|y| < h^{-1}$  and  $|\xi|^2 \in J_2$  on the support of the amplitude, we have

$$|\partial_\xi \Phi_h^+(\tau, x, y, \xi)| > c(|x| + h^{-1}) > c(1 + |x| + |y| + |\tau|), \quad 0 \leq \tau \leq h^{-1},$$

for some  $c > 0$  independent of  $h$ . Therefore, integrating by parts (3.6) with respect to

$$-ih|\partial_\xi \Phi_h^+|^{-2}(\partial_\xi \Phi_h^+) \cdot \partial_\xi,$$

we obtain

$$\begin{aligned} & \left| (2\pi h)^{-d} (1 - \rho(\varepsilon h x)) \partial_x^\alpha \partial_y^\beta \int e^{i\Phi_h^+(\tau, x, y, \xi)/h} b_h^+(x, \xi) \overline{c_h^+(y, \xi)} d\xi \right| \\ & \leq C_{\alpha\beta M} h^{M-d-|\alpha+\beta|} (1 + |x| + |y| + \tau)^{-M}, \end{aligned}$$

for all  $M \geq 0$ ,  $0 \leq \tau \leq h^{-1}$  and  $\alpha, \beta \in \mathbb{Z}_+^d$ . (3.5) then follows from the  $L^2$ -boundedness of FIOs.  $\square$

## 4 WKB parametrix

In the previous section we proved that  $e^{-ithH}$  is well approximated in terms of an Isozaki-Kitada parametrix on a time scale of order  $h^{-1}$  if we localize the initial data in regions  $\Gamma^\pm(R, J, \sigma) \cap \{x; R < |x| < h^{-1}\}$ . Therefore, it remains to control  $e^{-ithH}$  on a region  $\{x; |x| \gtrsim h^{-1}\}$ . In this section we construct the WKB parametrix for  $e^{-ithH}a(x, hD)$ , where  $a \in S(1, g)$  with  $\text{supp } a \subset \{(x, \xi) \in \mathbb{R}^{2d}; |x| \gtrsim h^{-1}, |\xi|^2 \in J\}$ . In what follows we assume that  $H$  satisfies Assumption A with  $\mu = 0$  and  $\nu = 1$ .

We first consider the phase function of the WKB parametrix, that is a solution to the time-dependent Hamilton-Jacobi equation generated by  $p_h(x, \xi) = k(x, \xi) + h^2 V(x)$ . For  $R > 0$  and open interval  $J \Subset (0, \infty)$ , we set

$$\Omega(R, J) := \{(x, \xi) \in \mathbb{R}^{2d}; |x| > R/2, |\xi|^2 \in J\}.$$

We note that  $\Omega(R_1, J_1) \subset \Omega(R_2, J_2)$  if  $R_1 > R_2$  and  $J_1 \subset J_2$ .

**Proposition 4.1.** *Choose arbitrarily an open interval  $J \Subset (0, \infty)$ . Then, there exist  $\delta_0 > 0$  and  $h_0 > 0$  small enough such that, for all  $h \in (0, h_0]$ ,  $0 < R \leq h^{-1}$  and  $0 < \delta \leq \delta_0$ , we can construct a family of smooth functions*

$$\{\Psi_h(t, x, \xi)\}_{h \in (0, h_0]} \subset C^\infty((-\delta R, \delta R) \times \mathbb{R}^{2d})$$

such that  $\Psi_h(t, x, \xi)$  satisfies the Hamilton-Jacobi equation associated to  $p_h$ :

$$\begin{cases} \partial_t \Psi_h(t, x, \xi) = -p_h(x, \partial_x \Psi_h(t, x, \xi)), & 0 < |t| < \delta R, (x, \xi) \in \Omega(R, J), \\ \Psi_h(0, x, \xi) = x \cdot \xi, & (x, \xi) \in \Omega(R, J). \end{cases} \quad (4.1)$$

Moreover, for all  $|t| \leq \delta R$  and  $\alpha, \beta \in \mathbb{Z}_+^d$ ,  $\Psi_h(t, x, \xi)$  satisfies

$$|\partial_x^\alpha \partial_\xi^\beta (\Psi_h(t, x, \xi) - x \cdot \xi)| \leq C \delta R^{1-|\alpha|}, \quad x, \xi \in \mathbb{R}^d, |\alpha + \beta| \geq 2, \quad (4.2)$$

$$|\partial_x^\alpha \partial_\xi^\beta (\Psi_h(t, x, \xi) - x \cdot \xi + t p_h(x, \xi))| \leq C_{\alpha\beta} \delta R^{|\alpha|} |t|, \quad x, \xi \in \mathbb{R}^d. \quad (4.3)$$

*Proof.* We give the proof in Appendix A.  $\square$

We next define the corresponding FIO. Let  $0 < R \leq h^{-1}$ ,  $J \Subset J_1 \Subset (0, \infty)$  open intervals and  $\Psi_h$  defined by the previous proposition with  $R, J$  replaced by  $R/4, J_1$ , respectively. Suppose that  $\{a_h(t, \cdot, \cdot)\}_{h \in (0, h_0], 0 \leq t \leq \delta R}$  is bounded in  $S(1, g)$  and supported in  $\Omega(R, J)$ . We then define the FIO for WKB parametrix  $\mathcal{F}_{\text{WKB}}(\Psi_h(t), a_h(t)) : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}^d)$  by

$$\mathcal{F}_{\text{WKB}}(\Psi_h(t), a_h(t))u(x) = \frac{1}{(2\pi h)^d} \int e^{i(\Psi_h(t, x, \xi) - y \cdot \xi)/h} a_h(t, x, \xi) u(y) dy d\xi.$$

**Lemma 4.2.**  $\mathcal{F}_{\text{WKB}}(\Psi_h(t), a(t))$  is bounded on  $L^2(\mathbb{R}^d)$  uniformly with respect to  $R, h$  and  $t$ :

$$\sup_{h \in (0, h_0], 0 \leq t \leq \delta R} \|\mathcal{F}_{\text{WKB}}(\Psi_h(t), a(t))\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C.$$

*Proof.* For  $|t| \leq \delta R$ , we define the map  $\tilde{\Xi}(t, x, \xi, y)$  on  $\mathbb{R}^{3d}$  by

$$\tilde{\Xi}(t, x, y, \xi) = \int_0^1 (\partial_x \Psi_h)(t, y + \lambda(x - y), \xi) d\lambda.$$

By (4.2),  $\tilde{\Xi}(t, x, y, \xi)$  satisfies

$$\left| \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma (\tilde{\Xi}(t, x, y, \xi) - \xi) \right| \leq C_{\alpha\beta\gamma} \delta R^{-|\alpha+\beta|}, \quad |t| \leq \delta R, \quad x, y \in \mathbb{R}^d,$$

and the map  $\xi \mapsto \tilde{\Xi}(t, x, \xi, y)$  hence is a diffeomorphism from  $\mathbb{R}^d$  onto itself for all  $|t| \leq \delta R$  and  $x, y \in \mathbb{R}^d$ , provided that  $\delta > 0$  is small enough. Let  $\xi \mapsto [\tilde{\Xi}]^{-1}(t, x, y, \xi)$  be the corresponding inverse.  $[\tilde{\Xi}]^{-1}$  satisfies the same estimate as that for  $\tilde{\Xi}$ :

$$\left| \partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma ([\tilde{\Xi}]^{-1}(t, x, y, \xi) - \xi) \right| \leq C_{\alpha\beta\gamma} \delta R^{-|\alpha+\beta|} \quad \text{on} \quad [-\delta R, \delta R] \times \mathbb{R}^{3d}.$$

Using the change of variables  $\xi \mapsto [\tilde{\Xi}]^{-1}$ ,  $\mathcal{F}_{\text{WKB}}(\Psi_h(t), a(t)) \mathcal{F}_{\text{WKB}}(\Psi_h(t), a(t))^*$  can be regarded as a semi-classical PDO with a smooth and bounded amplitude

$$a_h(t, x, [\tilde{\Xi}]^{-1}(t, x, y, \xi)) \overline{a_h(t, y, [\tilde{\Xi}]^{-1}(t, x, y, \xi))} |\det \partial_\xi [\tilde{\Xi}]^{-1}(t, x, y, \xi)|.$$

Therefore, the  $L^2$ -boundedness follows from the Calderón-Vaillancourt theorem.  $\square$

We now state the main result in this section.

**Theorem 4.3.** Let  $J \Subset J_0 \Subset J_1 \Subset (0, \infty)$  be open intervals. Then there exist  $\delta_0, h_0 > 0$  such that, for all  $h \in (0, h_0]$ ,  $0 < R \leq h^{-1}$ ,  $0 < \delta \leq \delta_0$  and all symbol

$$a \in S(1, g) \quad \text{with} \quad \text{supp } a \in \Omega(R, J),$$

and all  $N \geq 0$ , we can find a semi-classical symbol

$$b_h(t, x, \xi) = \sum_{j=0}^N h^j b_{h,j}(t, x, \xi)$$

with  $b_{h,j}(t, \cdot, \cdot)$  bounded in  $S(1, g)$  and  $\text{supp } b_{h,j}(t, \cdot, \cdot) \subset \Omega(R/2, J_0)$  uniformly with respect to  $h \in (0, h_0]$  and  $|t| \leq \delta R$  such that  $e^{-ithH} a(x, hD_x)$  can be brought to the form

$$e^{-ithH} a(x, hD_x) = \mathcal{F}_{\text{WKB}}(\Psi_h(t), b_h(t)) + Q_{\text{WKB}}(t, h, N),$$

where  $\mathcal{F}_{\text{WKB}}(\Psi_h(t), b_h(t))$  is the Fourier integral operator with the phase function  $\Psi_h(t, x, \xi)$ , defined in Proposition 4.1 with  $R, J$  replaced by  $R/4, J_1$  respectively, and its distribution kernel satisfies the dispersive estimates:

$$|K_{\text{WKB}}(t, h, x, y)| \leq C|th|^{-d/2}, \quad h \in (0, h_0], \quad 0 < |t| \leq \delta R, \quad x, \xi \in \mathbb{R}^d. \quad (4.4)$$

Moreover the remainder  $Q_{\text{WKB}}(t, h, N)$  satisfies

$$\|Q_{\text{WKB}}(t, h, N)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_N h^N |t|, \quad h \in (0, h_0], \quad |t| \leq \delta R.$$

Here the constants  $C, C_N > 0$  can be taken uniformly with respect to  $h, t$  and  $R$ .

**Remark 4.4.** The essential point of Theorem 4.3 is to construct the parametrix on the time interval  $|t| \leq \delta R$ . When  $|t| > 0$  is small and independent of  $R$ , such a parametrix construction is basically well known (cf. [19]).

*Proof of Theorem 4.3.* We consider the case when  $t \geq 0$  and the proof for  $t < 0$  is similar.

**Construction of the amplitude.** The Duhamel formula yields

$$\begin{aligned} & e^{-ithH} \mathcal{F}_{\text{WKB}}(\Psi_h(0), b_h(0)) \\ &= \mathcal{F}_{\text{WKB}}(\Psi_h(t), b_h(t)) + \frac{i}{h} \int_0^t e^{-i(t-s)hH} (hD_s + h^2 H) \mathcal{F}_{\text{WKB}}(\Psi_h(s), b_h(s)) ds. \end{aligned}$$

Therefore, it suffices to show that there exist  $b_{h,j}$  with  $b_{h,0}|_{t=0} = a$  and  $b_{h,j}|_{t=0} = 0$  for  $j \geq 1$  such that

$$\|(hD_s + h^2 H) \mathcal{F}_{\text{WKB}}(\Psi_h(s), b_h(s))\|_{\mathcal{L}(L^2)} \leq C_N h^{N+1}, \quad 0 \leq s \leq \delta R. \quad (4.5)$$

Let  $k + k_1$  be the full symbol of  $H_0$ :  $H_0 = k(x, D) + k_1(x, D)$ , and define a smooth vector field  $\mathcal{X}_h(t)$  and a function  $\mathcal{Y}_h(t)$  by

$$\mathcal{X}_h(t, x, \xi) := (\partial_\xi k)(x, \partial_x \Psi_h(t, x, \xi)), \quad \mathcal{Y}_h(t, x, \xi) := [(k + k_1)(x, \partial_x \Psi_h)(t, x, \xi)].$$

Symbols  $\{b_{h,j}\}$  can be constructed in terms of the method of characteristics as follows. For all  $0 \leq s, t \leq \delta R$ , we consider the flow  $z_h(t, s, x, \xi)$  generated by  $\mathcal{X}_h(t)$ , that is the solution to the following ODE:

$$\partial_t z_h(t, s, x, \xi) = \mathcal{X}_h(z_h(t, s, x, \xi), \xi); \quad z_h(s, s) = x.$$

Choose  $R', R''$  and two intervals  $J'_0, J''_0$  so that

$$R/2 > R' > R'' > R/4, \quad J_0 \Subset J'_0 \Subset J''_0 \Subset (0, \infty).$$

(4.3) and the same argument as that in the proof of Lemmas A.1 and A.2 imply that there exists  $\delta_0, h_0 > 0$  small enough such that, for all  $0 < \delta \leq \delta_0$ ,  $h \in (0, h_0]$ ,  $0 < R \leq h^{-1}$  and  $0 \leq s, t \leq \delta R$ ,  $z_h(t, s)$  is well defined on  $\Omega(R'', J''_0)$  and satisfies

$$|\partial_x^\alpha \partial_\xi^\beta (z_h(t, s, x, \xi) - x)| \leq C_{\alpha\beta} \delta R^{1-|\alpha|}. \quad (4.6)$$

In particular,  $(z_h(t, s, x, \xi), \xi) \in \Omega(R', J')$  for  $0 \leq s, t \leq \delta R$  if  $\delta > 0$ , depending only on  $J''$ , is small enough. We now define  $\{b_{h,j}(t, x, \xi)\}_{0 \leq j \leq N}$  inductively by

$$\begin{aligned} b_{h,0}(t, x, \xi) &= a(z_h(0, t), \xi) \exp \left( \int_0^t \mathcal{Y}_h(s, z_h(s, t, x, \xi), \xi) ds \right), \\ b_{h,j}(t, x, \xi) &= - \int_0^t (iH_0 b_{h,j-1})(s, z_h(s, t), \xi) \exp \left( \int_u^t \mathcal{Y}_h(u, z_h(u, t, x, \xi), \xi) du \right) ds. \end{aligned}$$

Since  $\text{supp } a \in \Omega(R, J)$  and  $z_h(t, s, \Omega(R, J)) \subset \{x; |x| > R/2\}$  for all  $0 \leq s, t \leq \delta R$ ,  $b_{h,j}(t)$  are supported in  $\Omega(R/2, J_0)$ . Thus, if we extend  $b_{h,j}$  on  $\mathbb{R}^{2d}$  so that

$$b_{h,j}(t, x, \xi) = 0, \quad (x, \xi) \notin \Omega(R/2, J_0),$$

then  $b_{h,j}$  is still smooth in  $(x, \xi)$ . By (4.3) and (4.6), we learn

$$|\partial_x^\alpha \partial_\xi^\beta \mathcal{Y}_h(s, z_h(s, t, x, \xi), \xi)| \leq C\delta R^{-1-|\alpha|}, \quad 0 \leq s, t \leq \delta R.$$

$\{b_{h,j}(t, \cdot, \cdot); h \in (0, h_0], 0 < R \leq h^{-1}, t \in [0, \delta R], 0 \leq j \leq N\}$  thus is a bounded set in  $S(1, g)$  and  $\text{supp } b_{h,j}(t, \cdot, \cdot) \subset \Omega(R/2, J_0)$  uniformly with respect to  $h \in (0, h_0]$  and  $0 \leq t \leq \delta R$ .

A standard Hamilton-Jacobi theory shows that  $b_{h,j}(t)$  satisfy the following transport equations:

$$\begin{cases} \partial_t b_{h,0}(t) + \mathcal{X}_h(t) b_{h,0}(t) + \mathcal{Y}_h(t) b_{h,0}(t) = 0, \\ \partial_t b_{h,j}(t) + \mathcal{X}_h(t) b_{h,j}(t) + \mathcal{Y}_h(t) b_{h,j}(t) = -iH_0 b_{h,j-1}(t), \quad j \geq 1, \end{cases} \quad (4.7)$$

with the initial condition  $b_{h,0}(0) = a$ ,  $b_{h,j}(0) = 0$ ,  $j = 1, 2, \dots, N$ . A direct computation then yields

$$e^{-i\Psi_h(s, x, \xi)/h} (hD_s + h^2 H) \left( e^{i\Psi_h(s, x, \xi)/h} \sum_{j=0}^N h^j b_{h,j} \right) = O(h^{N+1}) \text{ in } S(1, g)$$

which, combined with Lemma 4.2, implies (4.5).

**Dispersive estimates.** The distribution kernel of  $\mathcal{F}_{\text{WKB}}(\Psi_h(t), b_h(t))$  is given by

$$K_{\text{WKB}}(t, h, x, y) = \frac{1}{(2\pi h)^d} \int e^{\frac{i}{h}(\Psi_h(t, x, \xi) - y \cdot \xi)} b_h(t, x, \xi) d\xi.$$

Since  $b_h(t, x, \xi)$  has a compact support with respect to  $\xi$ ,

$$|K_{\text{WKB}}(t, h, x, y)| \leq Ch^{-d} \leq C|th|^{-d/2} \quad \text{for } 0 < t \leq h.$$

We hence assume  $h < t$  without loss of generality. Choose  $\chi \in S(1, g)$  so that  $0 \leq \chi \leq 1$ ,  $\chi \equiv 1$  on  $\Omega(R/2, J_0)$  and  $\text{supp } \chi \subset \Omega(R/4, J_1)$ , and set

$$\psi_h(t, x, y, \xi) = \frac{(x - y)}{t} \cdot \xi - p_h(x, \xi) + \chi(x, \xi) \left( \frac{\Psi_h(t, x, \xi) - x \cdot \xi}{t} + p_h(x, \xi) \right).$$

By the definition, we obtain

$$\psi_h(t, x, y, \xi) = \frac{\Psi_h(t, x, \xi) - y \cdot \xi}{t}, \quad t \in [h, \delta R], \quad (x, \xi) \in \Omega(R/2, J_1), \quad y \in \mathbb{R}^d,$$

and (4.3) implies

$$\left| \partial_x^\alpha \partial_\xi^\beta \psi_h(t, x, y, \xi) \right| \leq C_{\alpha\beta} \quad \text{on } [0, \delta R] \times \mathbb{R}^{3d}, \quad |\alpha + \beta| \geq 2.$$

Moreover,  $\partial_\xi^2 \psi_h(t, x, y, \xi)$  can be brought to the form

$$\partial_\xi^2 \psi_h(t, x, y, \xi) = -(a^{jk}(x))_{j,k} + Q_h(t, x, \xi),$$

where the error term  $Q_h(t, x, \xi)$  is a  $d \times d$ -matrix satisfying

$$\left| \partial_x^\alpha \partial_\xi^\beta Q_h(t, x, \xi) \right| \leq C_{\alpha\beta} \delta h^{|\alpha|} \quad \text{on } [0, \delta R] \times \mathbb{R}^{2d}.$$

Since  $(a^{jk}(x))$  is uniformly elliptic, the stationary phase theorem implies that

$$|K_{\text{WKB}}(t, h, x, y)| \leq Ch^{-d} |t/h|^{-d/2} = C|th|^{-d/2},$$

provided that  $\delta > 0$  is small enough. We complete the proof.

## 5 Proof of Theorem 1.1 (i)

In this section we complete the proof of Theorem 1.1 (i). Let  $\chi_0 \in C_0^\infty(\mathbb{R}^d)$  with  $\chi_0 \equiv 1$  on  $\{|x| < R_0\}$  and  $\psi \in C_0^\infty((0, \infty))$ . A partition unity argument and Lemma 2.1 show that there exist  $a^\pm \in S(1, g)$  with  $\text{supp } a^\pm \subset \Gamma^\pm(R_0, J, 1/2)$  such that  $(1 - \chi_0)\psi(h^2 H_0)$  is approximated by  $a^\pm(x, hD)$ :

$$(1 - \chi_0)\psi(h^2 H_0) = a^+(x, hD)^* + a^-(x, hD)^* + Q_0(h),$$

where  $J \Subset (0, \infty)$  is an open interval satisfying  $\pi_\xi(\text{supp } \varphi \circ k) \Subset J$  and  $Q_0(h)$  satisfies

$$\sup_{h \in (0, 1]} \|Q_0(h)\|_{\mathcal{L}(L^2(\mathbb{R}^d), L^q(\mathbb{R}^d))} \leq C_q, \quad q \geq 2.$$

Let  $b \in C_0^\infty(\mathbb{R}^d; \mathbb{R})$  be a cut-off function such that  $b \equiv 1$  on a neighborhood of  $J$ . By the asymptotic formula (2.1), we can write

$$a^\pm(x, hD)^* = b(hD)a^\pm(x, hD)^* + Q_1(h)$$

where  $Q_1(h)$  satisfies the same  $\mathcal{L}(L^2, L^q)$ -estimate as that of  $Q_0(h)$ . Therefore,

$$\|(Q_0(h) + Q_1(h))e^{-itH}u_0\|_{L^p([- \delta, \delta]; L^q(\mathbb{R}^d))} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}, \quad h \in (0, 1], \quad (5.1)$$

for any  $p, q \geq 2$ .

Next, we shall prove the following dispersive estimate for the main terms:

$$\begin{aligned} & \|b(hD)a^\pm(x, hD)^*e^{-i(t-s)H}a^\pm(x, hD)b(hD)\|_{\mathcal{L}(L^1(\mathbb{R}^d), L^\infty(\mathbb{R}^d))} \\ & \leq C|t-s|^{-d/2} \end{aligned} \quad (5.2)$$

for  $0 < |t-s| \leq \delta$ . We first consider the outgoing case. Let us fix  $N > 1$  so large that  $N \geq 2d+1$ . After rescaling  $t-s \mapsto (t-s)h$  and choosing  $R_0 > 1$  large enough, we apply Theorem 3.1 with  $R = R_0$ , Lemma 3.2 and Theorem 4.3 with  $R = h^{-1}$  to  $e^{-i(t-s)hH}a^+(x, hD)$ . Then, we can write

$$\begin{aligned} & e^{-i(t-s)hH}a^+(x, hD) \\ & = \mathcal{F}_{\text{IK}}(S_h^+, b_h^+)e^{i(t-s)h\Delta/2}\mathcal{F}_{\text{IK}}(S_h^+, c_h^+)^* + \mathcal{F}_{\text{WKB}}(\Psi_h(t-s), b_h(t-s)) \\ & \quad + Q_2^+(t-s, h), \end{aligned}$$

where the distribution kernels of main terms satisfy dispersive estimates

$$|K_{\text{IK}}^+(t-s, h, x, y)| + |K_{\text{WKB}}(t-s, h, x, y)| \leq C|(t-s)h|^{-d/2}, \quad (5.3)$$

uniformly with respect to  $h \in (0, h_0]$ ,  $0 < t-s \leq \delta h^{-1}$  and  $x, y \in \mathbb{R}^d$ . Let  $A(h, x, y)$  and  $B(h, x, y)$  be the distribution kernels of  $a(x, hD)^*$  and  $b(hD)$ , respectively. They clearly satisfy

$$\sup_x \int (|A(h, x, y)| + |B(h, x, y)|)dy + \sup_y \int (|A(h, x, y)| + |B(h, x, y)|)dx \leq C$$

uniformly in  $h \in (0, 1]$ . By using this estimate and (5.3), we see that the distribution kernel of  $b(hD)a^+(x, hD)^*(e^{-i(t-s)hH}a^+(x, hD) - Q_2^+(t-s, h))b(hD)$  satisfies the same dispersive estimates as (5.3) for  $0 < t-s \leq \delta h^{-1}$ . On the other hand,  $Q_2^+(t-s, h)$  satisfy

$$\|Q_2^+(t-s, h)\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leq C_N h^N, \quad h \in (0, h_0], \quad 0 \leq t-s \leq \delta h^{-1}.$$

We here recall that  $a^+(x, hD)^*$  is uniformly bounded on  $L^2(\mathbb{R}^d)$  in  $h \in (0, 1]$  and  $b(hD)$  satisfies

$$\begin{aligned} & \|b(hD)\|_{\mathcal{L}(H^{-s}(\mathbb{R}^d), H^s(\mathbb{R}^d))} \\ & \leq \| \langle D \rangle^s \langle hD \rangle^{-s} \|_{\mathcal{L}(L^2(\mathbb{R}^d))} \| \langle hD \rangle^s b(hD) \langle hD \rangle^s \|_{\mathcal{L}(L^2(\mathbb{R}^d))} \| \langle hD \rangle^{-s} \langle D \rangle^s \|_{\mathcal{L}(L^2(\mathbb{R}^d))} \\ & \leq C_s h^{-2s}. \end{aligned}$$

$b(hD)a^+(x, hD)^*Q_2^+(t-s, h)b(hD)$  hence is a bounded operator in  $\mathcal{L}(H^{-s}, H^s)$  for some  $s > d/2$ . Its distribution kernel  $\tilde{Q}_2^+(t-s, h, x, y)$  thus is uniformly bounded on  $\mathbb{R}^{2d}$  with respect to  $h \in (0, h_0]$  and  $0 \leq t-s \leq \delta h^{-1}$ . Therefore,

$$|\tilde{Q}_2^+(t-s, h, x, y)| \lesssim 1 \lesssim |(t-s)h|^{-d/2}, \quad h \in (0, h_0], \quad 0 < t-s \leq \delta h^{-1}.$$

The corresponding estimates for the incoming case also hold for  $0 \leq -(t-s) \leq \delta h^{-1}$ . Therefore,  $b(hD)a^\pm(x, hD)^*e^{-i(t-s)hH}a^\pm(x, hD)b(hD)$  have distribution kernels  $K^\pm(t-s, h, x, y)$  satisfying

$$|K^\pm(t-s, h, x, y)| \leq C|(t-s)h|^{-d/2} \quad (5.4)$$

uniformly with respect to  $h \in (0, h_0]$ ,  $0 \leq \pm(t-s) \leq \delta h^{-1}$  and  $x, y \in \mathbb{R}^d$ , respectively.

We here use a simple trick due to Bouclet-Tzvetkov [2, Lemma 4.3.]. If we set  $U^\pm(t, h) = b(hD)a^\pm(x, hD)^*e^{-itH}a^\pm(x, hD)b(hD)$ , then

$$U^\pm(s-t, h) = U^\pm(t-s, h)^*,$$

and hence  $K^\pm(s-t, h, x, y) = \overline{K^\pm(t-s, h, y, x)}$ . Therefore, the estimates (5.4) also hold for  $0 < \mp(t-s) \leq \delta h^{-1}$  and  $x, y \in \mathbb{R}^d$ . Rescaling  $(t-s)h \mapsto t-s$ , we obtain the estimate (5.2).

Finally, since the  $\mathcal{L}(L^2)$ -boundedness of  $a^\pm(x, hD)^*e^{-itH}$  is obvious, (5.1), (5.2) and the Keel-Tao theorem [15] imply the desired semi-classical Strichartz estimates:

$$\sup_{h \in (0, h_0]} \|(1 - \chi_0)\psi_0(h^2 H_0)e^{-itH}u_0\|_{L^p([- \delta, \delta]; L^q(\mathbb{R}^d))} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}.$$

By the virtue of Proposition 2.4, we complete the proof of Theorem 1.1 (i).

## 6 Proof of Theorem 1.1 (ii)

In this section we prove Theorem 1.1 (ii). Suppose that  $H$  satisfies Assumption A with  $\mu = \nu = 0$ . We first recall the local smoothing effects for Schrödinger operators with at most quadratic potentials proved by Doi [9]. For any  $s \in \mathbb{R}$ , we set  $\mathcal{B}^s := \{f \in L^2(\mathbb{R}^d); \langle x \rangle^s f \in L^2(\mathbb{R}^d), \langle D \rangle^s f \in L^2(\mathbb{R}^d)\}$ , and define a symbol  $e_s$  by

$$e_s(x, \xi) := (k(x, \xi) + |x|^2 + L(s))^{s/2} \in S((1 + |x| + |\xi|)^s, g).$$

We denote by  $E_s$  its Weyl quantization:

$$E_s f(x) = \frac{1}{2\pi} \int e^{i(x-y) \cdot \xi} e_s\left(\frac{x+y}{2}, \xi\right) f(y) dy d\xi.$$

Here  $L(s) > 1$  is a large constant depending on  $s$ . Then, for any  $s \in \mathbb{R}$ , there exists  $L(s) > 0$  such that  $E_s$  is a homeomorphism from  $\mathcal{B}^{r+s}$  to  $\mathcal{B}^r$  for all  $r \in \mathbb{R}$ , and  $(E_s)^{-1}$  is still a Weyl quantization of a symbol in  $S((1 + |x| + |\xi|)^{-s}, g)$ .

**Lemma 6.1** (The local smoothing effects [9]). *Suppose that the kinetic energy  $k(x, \xi)$  satisfies the non-trapping condition (1.5). Then, for any  $T > 0$  and  $\sigma > 0$ , there exists  $C_T > 0$  such that*

$$\|\langle x \rangle^{-1/2-\sigma} E_{1/2} u\|_{L^2([-T, T]; L^2(\mathbb{R}^d))} \leq C_T \|u_0\|_{L^2}, \quad (6.1)$$

where  $u = e^{-itH}u_0$ .

**Remark 6.2.** Let  $\chi \in C_0^\infty(\mathbb{R}^d)$ . (6.1) implies a usual local smoothing effect:

$$\|\langle D \rangle^{1/2} \chi u\|_{L^2([-T, T]; L^2(\mathbb{R}^d))} \leq C_T \|u_0\|_{L^2(\mathbb{R}^d)}. \quad (6.2)$$



Indeed, let  $\chi_1 \in C_0^\infty(\mathbb{R}^d)$  be such that  $\chi_1 \equiv 1$  on  $\text{supp } \chi$ . We split  $\langle D \rangle^{1/2} \chi$  as follows:

$$\begin{aligned} \langle D \rangle^{1/2} \chi &= \chi_1 \langle D \rangle^{1/2} \chi + [\langle D \rangle^{1/2}, \chi_1] \chi, \\ \chi_1 \langle D \rangle^{1/2} \chi &= \chi_1 \langle D \rangle^{1/2} (E_{1/2})^{-1} E_{1/2} \chi \\ &= \chi_1 \langle D \rangle^{1/2} (E_{1/2})^{-1} \chi_1 E_{1/2} \chi + \chi_1 \langle D \rangle^{1/2} (E_{1/2})^{-1} [E_{1/2}, \chi_1] \chi. \end{aligned}$$

By a standard symbolic calculus,  $[\langle D \rangle^{1/2}, \chi_1] \chi$ ,  $\chi_1 \langle D \rangle^{1/2} (E_{1/2})^{-1}$  and  $[E_{1/2}, \chi_1] \chi$  are bounded on  $L^2(\mathbb{R}^d)$  since  $\chi_1$  has a compact support. Therefore, Lemma 6.1 implies

$$\begin{aligned} \|\langle D \rangle^{1/2} \chi u\|_{L^2([-T, T]; L^2(\mathbb{R}^d))} &\leq C \|\chi_1 E_{1/2} \chi u\|_{L^2([-T, T]; L^2(\mathbb{R}^d))} + C_T \|u\|_{L^2(\mathbb{R}^d)} \\ &\leq C_T \|u_0\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

*Proof of Theorem 1.1 (ii).* We consider the case when  $0 \leq t \leq T$  only, and the proof for the negative time is similar. We mimic the argument in [18, Section II.2]. A direct computation yields

$$\begin{aligned} (i\partial_t + \Delta) \chi u &= \Delta \chi u + \chi H u \\ &= \chi_1 (H + \Delta) \chi_1 \chi u + (\chi_1 [\chi, H] + [\Delta, \chi_1] \chi) u. \end{aligned}$$

We define a self-adjoint operator by  $\tilde{H} := -\Delta + \chi_1 (H + \Delta) \chi_1$ , and set

$$\tilde{U}(t) := e^{-it\tilde{H}}, \quad F := (\chi_1 [\chi, H] + [\Delta, \chi_1] \chi) u.$$

We here note that if  $H_0$  satisfies the non-trapping condition then so does the principal part of  $\tilde{H}$ . By the Duhamel formula, we can write

$$\chi u = \tilde{U}(t) \chi u_0 + \int_0^t \tilde{U}(t-s) F(s) ds.$$

Since  $\chi_1 (H + \Delta) \chi_1$  is a compactly supported smooth perturbation, it was proved by Staffilani-Tataru [22] that  $\tilde{U}(t)$  is bounded from  $L^2(\mathbb{R}^d)$  to  $L^2([0, T]; H_{loc}^{1/2}(\mathbb{R}^d))$ , and that its adjoint

$$\tilde{U}^* f = \int_0^T U(-s) f(s, \cdot) ds$$

is bounded from  $L^2([0, T]; H_{loc}^{-1/2}(\mathbb{R}^d))$  to  $L^2(\mathbb{R}^d)$ . Moreover,  $\tilde{U}(t)$  satisfies Strichartz estimates (for any admissible pair  $(p, q)$ ):

$$\|\tilde{U}(t) v\|_{L^p([-T, T]; L^q(\mathbb{R}^d))} \leq C_T \|v\|_{L^2},$$

Therefore, we have

$$\begin{aligned} \left\| \int_0^T \tilde{U}(t-s) F(s) ds \right\|_{L^p([-T, T]; L^q(\mathbb{R}^d))} &\leq C_T \|U^* F\|_{L^2(\mathbb{R}^d)} \\ &\leq C_T \|\langle D \rangle^{-1/2} F\|_{L^2([-T, T]; L^2(\mathbb{R}^d))} \end{aligned}$$

since  $F$  has a compact support with respect to  $x$ . The Christ-Kiselev lemma (see [7, 21]) then implies

$$\left\| \int_0^t \tilde{U}(t-s) F(s) ds \right\|_{L^p([-T, T]; L^q(\mathbb{R}^d))} \leq C_T \|\langle D \rangle^{-1/2} F\|_{L^2([-T, T]; L^2(\mathbb{R}^d))},$$

provided that  $p > 2$ . We split  $F$  as

$$F = ([\chi, H] \chi_1 + [\Delta, \chi_1] \chi) u + [\chi_1, [\chi, H]] u =: F_1 + F_2.$$

Since  $[\chi, H]$  is a first order differential operator with bounded coefficients, we see that  $[\chi_1, [\chi, H]]$  is bounded on  $L^2(\mathbb{R}^d)$ , and  $\|\langle D \rangle^{-1/2} F_2\|_{L^2([-T, T]; L^2(\mathbb{R}^d))}$  is dominated by  $C_T \|u_0\|_{L^2(\mathbb{R}^d)}$ . We now use (6.2) and obtain

$$\begin{aligned} \|\langle D \rangle^{-1/2} F_1\|_{L^2([-T, T]; L^2(\mathbb{R}^d))} &\leq C \|\chi_1 u\|_{L^2([-T, T]; H^{-1/2}(\mathbb{R}^d))} \\ &\leq C \|\langle D \rangle^{1/2} \chi_1 u\|_{L^2([-T, T]; L^2(\mathbb{R}^d))} \\ &\leq C_T \|u_0\|_{L^2}, \end{aligned}$$

which completes the proof.

## A Proof of Proposition 4.1

Assume Assumption A with  $\mu = 0$ ,  $\nu \geq 0$ . We here give the detail of the proof of Proposition 4.1. We first study the corresponding classical mechanics. Consider the Hamilton flow

$$(X_h(t), \Xi_h(t)) = (X_h(t, x, \xi), \Xi_h(t, x, \xi)), \quad h \in (0, 1],$$

generated by the semi-classical total energy

$$p_h(x, \xi) = k(x, \xi) + h^2 V(x),$$

i.e.,  $(X_h(t), \Xi_h(t))$  is the solution to the Hamilton equations

$$\begin{cases} \dot{X}_{h,j}(t) = \sum_k a^{jk}(X_h(t)) \Xi_{h,k}(t), \\ \dot{\Xi}_{h,j}(t) = -\frac{1}{2} \sum_{k,l} \frac{\partial a^{kl}}{\partial x_j}(X_h(t)) \Xi_{h,k}(t) \Xi_{h,l}(t) - h^2 \frac{\partial V}{\partial x_j}(X_h(t)), \end{cases}$$

with the initial condition  $(X_h(0), \Xi_h(0)) = (x, \xi)$ , where  $\dot{f} = \partial_t f$ . We first prepare an a priori bound of the flow.

**Lemma A.1.** *For all  $h \in (0, 1]$ ,  $|t| \lesssim h^{-1}$  and  $(x, \xi) \in \mathbb{R}^{2d}$ ,*

$$|X_h(t) - x| \lesssim (|\xi| + h\langle x \rangle^{1-\nu/2}) |t|, \quad |\Xi_h(t)| \lesssim |\xi| + h\langle x \rangle^{1-\nu/2}.$$

*Proof.* We consider the case  $t \geq 0$ . The proof for the case  $t < 0$  is analogous. Since the Hamilton flow conserves the total energy, namely

$$p_h(x, \xi) = p_h(X_h(t), \Xi_h(t)) \quad \text{for all } t \in \mathbb{R},$$

we have

$$\begin{aligned} |\Xi_h(t)| &\lesssim \sqrt{p_0(X_h(t), \Xi_h(t))} \\ &\lesssim \sqrt{p_h(x, \xi) - h^2 V(X_h(t))} \\ &\lesssim |\xi| + h\langle x \rangle^{1-\nu/2} + h\langle X_h(t) \rangle^{1-\nu/2}. \end{aligned}$$

Applying the above inequality to the Hamilton equation, we have

$$|\dot{X}^h(t)| \lesssim |\Xi_h(t)| \lesssim |\xi| + h\langle x \rangle^{1-\nu/2} + h\langle X_h(t) \rangle^{1-\nu/2}.$$

Integrating with respect to  $t$  and using Gronwall's inequality, we obtain the assertion since  $e^{th} \lesssim |t|$  for  $|t| \lesssim h^{-1}$ .  $\square$

Let  $J \Subset (0, \infty)$  be an open interval. For sufficiently small  $\delta > 0$  and for all  $0 < R \leq h^{-1}$ , the above lemma implies

$$|x|/2 \leq |X_h(t, x, \xi)| \leq 2|x| \quad (\text{A.1})$$

uniformly with respect to  $h \in (0, 1]$ ,  $|t| \leq \delta R$  and  $(x, \xi) \in \Omega(R, J)$ . By using this inequality, we have the following:

**Lemma A.2.** *Let  $J, \delta$  be as above. Then, for  $h \in (0, 1]$ ,  $0 < R \leq h^{-1}$ ,  $|t| \leq \delta R$  and  $(x, \xi) \in \Omega(R, J)$ ,  $X_h(t, x, \xi)$  and  $\Xi_h(t, x, \xi)$  satisfy*

$$\begin{cases} |X_h(t) - x| \leq C(1 + \delta h \langle x \rangle^{1-\nu})|t|, \\ |\Xi_h(t) - \xi| \leq C(\langle x \rangle^{-1} + h^2 \langle x \rangle^{1-\nu})|t|, \end{cases} \quad (\text{A.2})$$

and, for  $|\alpha + \beta| = 1$ ,

$$\begin{cases} |\partial_x^\alpha \partial_\xi^\beta (X_h(t) - x)| \leq C_{\alpha\beta} \left( \langle x \rangle^{-|\alpha|} + h^{|\alpha|} \langle x \rangle^{-|\alpha|\nu/2} \right) |t|, \\ |\partial_x^\alpha \partial_\xi^\beta (\Xi_h(t) - \xi)| \leq C_{\alpha\beta} \left( \langle x \rangle^{-1-|\alpha|} + h^{1+|\alpha|} \langle x \rangle^{-(1+|\alpha|)\nu/2} \right) |t|, \end{cases} \quad (\text{A.3})$$

and, for  $|\alpha + \beta| \geq 2$ ,

$$\begin{cases} |\partial_x^\alpha \partial_\xi^\beta (X_h(t) - x)| \leq C_{\alpha\beta} \delta h^{|\alpha|} \langle x \rangle^{-1} R |t|, \\ |\partial_x^\alpha \partial_\xi^\beta (\Xi_h(t) - \xi)| \leq C_{\alpha\beta} h^{|\alpha|} \langle x \rangle^{-1} |t|. \end{cases} \quad (\text{A.4})$$

Moreover  $C, C_{\alpha\beta} > 0$  may be taken uniformly with respect to  $R, h$  and  $t$ .

*Proof.* We only prove the case when  $t \geq 0$ , the proof for the case  $t \leq 0$  is similar. Applying Lemma A.1 and (A.1) to the Hamilton equation, we have

$$\begin{aligned} |\dot{\Xi}^h(t)| &\lesssim \langle X_h(t) \rangle^{-1} |\Xi_h(t)|^2 + h^2 \langle X_h(t) \rangle^{1-\nu} \\ &\lesssim \langle x \rangle^{-1} (1 + h^2 \langle x \rangle^{2-\nu}) + h^2 \langle x \rangle^{1-\nu} \\ &\lesssim \langle x \rangle^{-1} + h^2 \langle x \rangle^{1-\nu}, \\ |\dot{X}^h(t)| &\lesssim |\Xi_h(t)| \lesssim 1 + \delta h \langle x \rangle^{1-\nu}, \end{aligned}$$

and (A.2) follows.

We next prove (A.3). By differentiating the Hamilton equation with respect to  $\partial_x^\alpha \partial_\xi^\beta$ ,  $|\alpha + \beta| = 1$ , we have

$$\frac{d}{dt} \begin{pmatrix} \partial_x^\alpha \partial_\xi^\beta X_h \\ \partial_x^\alpha \partial_\xi^\beta \Xi_h \end{pmatrix} = \begin{pmatrix} \partial_x \partial_\xi p_h(X_h, \Xi_h) & \partial_\xi^2 p_h(X_h, \Xi_h) \\ -\partial_x^2 p_h(X_h, \Xi_h) & -\partial_\xi \partial_x p_h(X_h, \Xi_h) \end{pmatrix} \begin{pmatrix} \partial_x^\alpha \partial_\xi^\beta X_h \\ \partial_x^\alpha \partial_\xi^\beta \Xi_h \end{pmatrix}. \quad (\text{A.5})$$

Define a weight function  $w_h(x) = \langle x \rangle^{-1} + h \langle x \rangle^{-\nu/2}$ . A direct computation and (A.2) then imply

$$\begin{aligned} \left| (\partial_x^\alpha \partial_\xi^\beta p_h)(X_h(t), \Xi_h(t)) \right| &\leq C_{\alpha\beta} w_h(x)^{|\alpha|}, \quad |\alpha + \beta| = 2, \\ \left| (\partial_x^\alpha \partial_\xi^\beta p_h)(X_h(t), \Xi_h(t)) \right| &\leq C_{\alpha\beta} \langle x \rangle^{2-|\alpha+\beta|} w_h(x)^{|\alpha|-1}, \quad |\alpha + \beta| \geq 3, \end{aligned}$$

for all  $|t| \leq \delta R$  and  $(x, \xi) \in \Omega(R, J)$ , and  $\partial_\xi^\beta p_h \equiv 0$  on  $\mathbb{R}^{2d}$  for  $|\beta| \geq 3$ . By integrating (A.5) with respect to  $t$ , we have

$$\begin{aligned} &w_h(x) |\partial_x^\alpha \partial_\xi^\beta (X_h(t) - x)| + |\partial_x^\alpha \partial_\xi^\beta (\Xi_h(t) - \xi)| \\ &\lesssim \int_0^t \left( w_h(x) \left( w_h(x) |\partial_x^\alpha \partial_\xi^\beta (X_h(t) - x)| + |\partial_x^\alpha \partial_\xi^\beta (\Xi_h(t) - \xi)| \right) + w_h(x)^{1+|\alpha|} \right) d\tau \end{aligned}$$

Using Gronwall's inequality, we have (A.3) since  $|t| \leq \delta R$ .

For  $|\alpha + \beta| \geq 2$ , we shall prove the estimate for  $\partial_{\xi_1}^2 X_h(t)$  only. Proofs for other cases are similar, and for higher derivatives follow from an induction on  $|\alpha + \beta|$ . By the Hamilton equation and (A.3), we learn

$$\partial_{\xi_1}^2 X_h = \partial_x \partial_{\xi} p_h(X_h, \Xi_h) \partial_{\xi_1}^2 X_h + \partial_{\xi}^2 p_h(X_h, \Xi_h) \partial_{\xi_1}^2 \Xi_h + Q(h, x, \xi)$$

where  $Q(h, x, \xi)$  satisfies

$$\begin{aligned} Q(h, x, \xi) &\leq C \sum_{|\alpha+\beta|=3, |\beta|=1,2} (\partial_x^\alpha \partial_{\xi}^\beta p)(X_h, \Xi_h) (\partial_{\xi_1} X_h)^{|\alpha|} (\partial_{\xi_1} \Xi_h)^{|\beta|} \\ &\leq C \langle x \rangle^{-1} \sum_{|\alpha|=1,2,3} w_h(x)^{|\alpha|-1} |t|^{|\alpha|} \\ &\leq C \delta \langle x \rangle^{-1} R. \end{aligned}$$

We similarly obtain

$$\partial_{\xi_1}^2 \Xi_h = -\partial_x^2 p_h(X_h, \Xi_h) \partial_{\xi_1}^2 X_h - \partial_{\xi} \partial_x p_h(X_h, \Xi_h) \partial_{\xi_1}^2 \Xi_h + O(\langle x \rangle^{-1}),$$

and these estimates and Gronwall's inequality imply

$$\begin{aligned} &(\delta R)^{-1} |\partial_{\xi_1}^2 X_h(t)| + |\partial_{\xi_1}^2 \Xi_h(t)| \\ &\lesssim \int_0^t w_h(x) \left( (\delta R)^{-1} |\partial_{\xi_1}^2 X_h(\tau)| + |\partial_{\xi_1}^2 \Xi_h(\tau)| \right) + \langle x \rangle^{-1} d\tau \\ &\lesssim \langle x \rangle^{-1} |t| \end{aligned}$$

for  $0 \leq t \leq \delta R$ . We hence have the assertion.  $\square$

**Remark A.3.** If  $\nu = 1$ , then Lemma A.2 implies that for any  $\alpha, \beta \in \mathbb{Z}_+^d$ , there exists  $C_{\alpha\beta}$  such that

$$|\partial_x^\alpha \partial_{\xi}^\beta (X_h(t) - x)| \leq C_{\alpha\beta} \delta R^{1-|\alpha|}, \quad |\partial_x^\alpha \partial_{\xi}^\beta (\Xi_h(t) - \xi)| \leq C_{\alpha\beta} \delta R^{-|\alpha|}, \quad (\text{A.6})$$

uniformly with respect to  $h \in (0, 1]$ ,  $0 < R \leq h^{-1}$ ,  $|t| \leq \delta R$  and  $(x, \xi) \in \Omega(R, J)$ .

**Lemma A.4.** Suppose that  $\nu = 1$  and let  $J_1 \Subset J'_1 \Subset (0, \infty)$  be open intervals. Then there exists  $\delta > 0$  small enough such that, for any fixed  $|t| \leq \delta R$ , the map

$$g_h(t) : (x, \xi) \mapsto (X_h(t, x, \xi), \xi)$$

is a diffeomorphism from  $\Omega(R/2, J'_1)$  onto its range. Moreover, we have

$$\Omega(R, J_1) \subset g^h(t, \Omega(R/2, J'_1)), \quad |t| \leq \delta R. \quad (\text{A.7})$$

*Proof.* We choose  $J''_1$  so that  $J'_1 \Subset J''_1 \Subset (0, \infty)$ . Choosing  $\chi \in S(1, g)$  such that

$$0 \leq \chi \leq 1, \quad \text{supp } \chi \subset \Omega(R/3, J''_1), \quad \chi \equiv 1 \text{ on } \Omega(R/2, J'_1),$$

we define  $X_h^\chi(t, x, \xi) := (1 - \chi(x, \xi))x + \chi(x, \xi)X_h(t, x, \xi)$  and set

$$g_h^\chi(t, x, \xi) = (X_h^\chi(t, x, \xi), \xi).$$

We also define  $(z, \xi) \mapsto \tilde{g}_h^\chi(t, z, \xi)$  by

$$\tilde{g}_h^\chi(t, z, \xi) = (\tilde{X}_h^\chi(t, z, \xi), \xi) := (X_h^\chi(t, Rz, \xi)/R, \xi).$$

By (A.6), there exists  $\delta > 0$  so small that, for  $|t| \leq \delta R$ ,  $(z, \xi) \in \mathbb{R}^{2d}$ ,

$$|\partial_z^\alpha \partial_\xi^\beta (\tilde{X}_h^\chi(t, z, \xi) - z)| \lesssim \delta R^{-|\alpha|}, \quad |\partial_z^\alpha \partial_\xi^\beta (J(\tilde{g}_h^\chi)(t, z, \xi) - \text{Id})| \leq C_{\alpha\beta} \delta < 1/2,$$

where  $J(\tilde{g}_h^\chi)$  is the Jacobi matrix with respect to  $(z, \xi)$ . The Hadamard global inverse mapping theorem then shows that  $\tilde{g}_h^\chi(t)$  is a diffeomorphism from  $\mathbb{R}^{2d}$  onto itself if  $|t| \leq \delta R$ . By definition,  $g_h(t)$  is a diffeomorphism from  $\Omega(R/2, J_1')$  onto its range.

We next prove (A.7). Since  $g_h(t) = g_h^\chi(t)$  and  $g_h^\chi(t)$  is bijective on  $\Omega(R/2, J_1')$ , it suffices to check that

$$\Omega(R, J_1)^c \supset g_h^\chi(t, \Omega(R/2, J_1')^c).$$

Suppose that  $(x, \xi) \in \Omega(R/2, J_1')^c$ . If  $(x, \xi) \in \Omega(R/3, J_1'')^c$ , then

$$g_h^\chi(t, x, \xi) = (x, \xi) \in \Omega(R/3, J_1'')^c \subset \Omega(R, J_1)^c.$$

Suppose that  $(x, \xi) \in \Omega(R/3, J_1'') \setminus \Omega(R/2, J_1')$ . By (A.2) and the support property of  $\chi$ , we have

$$|X_h^\chi(t)| \leq |x| + |\chi(X_h(t) - x)| \leq R/2 + C\delta R$$

for some  $C > 0$  independent of  $R$  and  $h$ . Choosing  $\delta$  satisfying  $1/2 + C\delta < 1$ , we obtain  $g_h^\chi(t, x, \xi) \in \Omega(R, J_1)^c$ .  $\square$

Let  $\Omega(R, J_1) \ni (x, \xi) \mapsto (Y_h(t, x, \xi), \xi)$  be the inverse of  $\Omega(R/2, J_1') \ni (x, \xi) \mapsto (X_h(t, x, \xi), \xi)$ .

**Lemma A.5.** *Let  $\delta, J_1$  as above and  $\nu = 1$ . Then, for all  $h \in (0, 1]$ ,  $0 < R \leq h^{-1}$ ,  $0 < |t| \leq \delta R$  and  $(x, \xi) \in \Omega(R, J_1)$ , we have*

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta (Y_h(t, x, \xi) - x)| &\leq C_{\alpha\beta} \delta R^{1-|\alpha|}, \\ |\partial_x^\alpha \partial_\xi^\beta (\Xi_h(t, Y_h(t, x, \xi)) - \xi)| &\leq C_{\alpha\beta} \delta R^{-|\alpha|}. \end{aligned}$$

*Proof.* We prove the inequalities for  $Y_h$  only. Proofs for  $\Xi_h(t, Y_h(t, x, \xi), \xi)$  are similar. Since  $(Y_h(t, x, \xi), \xi) \in \Omega(R/2, J_1')$ ,

$$\begin{aligned} |Y_h(t, x, \xi) - x| &= |X_h(0, Y_h(t, x, \xi), \xi) - X_h(t, Y_h(t, x, \xi), \xi)| \\ &\leq \sup_{(x, \xi) \in \Omega(R/2, J_1')} |X_h(t, x, \xi) - x| \\ &\lesssim \delta R. \end{aligned}$$

Next, let  $\alpha, \beta \in \mathbb{Z}_+^d$  with  $|\alpha| + |\beta| = 1$  and apply  $\partial_x^\alpha \partial_\xi^\beta$  to the equality

$$x = X_h(t, Y_h(t, x, \xi), \xi).$$

We then have the following equality

$$A(t, Z_h(t)) \partial_x^\alpha \partial_\xi^\beta (Y_h(t, x, \xi) - x) = \partial_y^\alpha \partial_\eta^\beta (y - X_h(t, y, \eta))|_{(y, \eta) = Z_h(t)}, \quad (\text{A.8})$$

where  $Z_h(t, x, \xi) = (Y_h(t, x, \xi), \xi)$  and  $A(t, Z) = (\partial_x X_h)(t, Z)$ . By (A.2) and a similar argument as that in the proof of Lemma A.4, we learn that  $A(Z^h(t))$  is invertible, and that  $A(Z^h(t))$  and  $A(Z^h(t))^{-1}$  are uniformly bounded with respect to  $h \in (0, 1]$ ,  $|t| \leq \delta R$  and  $(x, \xi) \in \Omega(R, J_1)$ . Therefore,

$$\begin{aligned} \left| \partial_x^\alpha \partial_\xi^\beta (Y_h(t, x, \xi) - x) \right| &\leq \sup_{(x, \xi) \in \Omega(R/2, J_1')} \left| \partial_y^\alpha \partial_\eta^\beta (y - X_h(t, y, \eta)) \right| \\ &\leq C_{\alpha\beta} \delta R^{1-|\alpha|}. \end{aligned}$$

The proof for higher derivatives is obtained by an induction on  $|\alpha| + |\beta|$ , and we omit the details.  $\square$

*Proof of Proposition 4.1.* We consider the case when  $t \geq 0$ , and the proof for  $t \leq 0$  is similar. Choosing  $J \in J_1 \Subset (0, \infty)$ , we define the action integral  $\tilde{\Psi}_h(t, x, \xi)$  on  $[0, \delta R] \times \Omega(R/2, J_1)$  by

$$\tilde{\Psi}_h(t, x, \xi) := x \cdot \xi + \int_0^t L_h(X_h(s, Y_h(t, x, \xi), \xi), \Xi_h(s, Y_h(t, x, \xi), \xi)) ds,$$

where  $L_h(x, \xi) = \xi \cdot \partial_\xi p_h(x, \xi) - p_h(x, \xi)$  is the Lagrangian associated to  $p_h$  and  $Y_h$  is defined by the above argument with  $R > 0$  replaced by  $R/2$ . The smoothness property of  $\tilde{\Psi}_h$  follows from corresponding properties of  $X_h$ ,  $\Xi_h$  and  $Y_h$ . By the standard Hamilton-Jacobi theory,  $\tilde{\Psi}_h(t, x, \xi)$  solves the Hamilton-Jacobi equation (4.1) on  $\Omega(R/2, J_1)$  and satisfies

$$\partial_x \tilde{\Psi}_h(t, x, \xi) = \Xi_h(t, Y_h(t, x, \xi), \xi), \quad \partial_\xi \tilde{\Psi}_h(t, x, \xi) = Y_h(t, x, \xi).$$

In particular, we obtain the following energy conservation law:

$$p_h(x, \partial_x \tilde{\Psi}_h(t, x, \xi)) = p_h(Y_h(t, x, \xi), \xi).$$

This energy conservation and Lemma A.5 imply

$$\begin{aligned} & |p_h(\partial_x \tilde{\Psi}_h(t, x, \xi) - p_h(x, \xi))| \\ & \leq |Y_h(t, x, \xi) - x| \int_0^1 |\partial_x p_h(\lambda x + (1 - \lambda)Y_h(t, x, \xi), \xi)| d\lambda \\ & \leq C\delta R(\langle x \rangle^{-1} + h^2) \\ & \leq C\delta. \end{aligned}$$

By using Lemma A.5, we also obtain

$$|\partial_x^\alpha \partial_\xi^\beta (p_h(x, \partial_x \tilde{\Psi}_h(t, x, \xi)) - p_h(x, \xi))| \leq C_{\alpha\beta} \delta R^{|\alpha|}, \quad \alpha, \beta \in \mathbb{Z}_+^d.$$

Therefore,

$$\left| \partial_x^\alpha \partial_\xi^\beta \left( \tilde{\Psi}_h(t, x, \xi) - x \cdot \xi + t p_h(x, \xi) \right) \right| \leq C_{\alpha\beta} \delta R^{|\alpha|} |t|.$$

Choose  $\chi \in S(1, g)$  so that

$$0 \leq \chi \leq 1, \quad \chi \equiv 1 \text{ on } \Omega(R, J) \text{ and } \text{supp } \chi \subset \Omega(R/2, J_1),$$

and define

$$\Psi_h(t, x, \xi) := x \cdot \xi - t p_h(x, \xi) + \chi(x, \xi) (\tilde{\Psi}_h(t, x, \xi) - x \cdot \xi + t p_h(x, \xi)).$$

Clearly,  $\Psi_h(t, x, \xi)$  satisfies the statement of Proposition 4.1.

## References

- [1] Bouclet, J.-M.: Strichartz estimates on asymptotically hyperbolic manifolds. *Anal. PDE.* **4**, 1–84 (2011)
- [2] Bouclet, J.-M., Tzvetkov, N.: Strichartz estimates for long range perturbations. *Amer. J. Math.* **129**, 1565–1609 (2007)
- [3] Bouclet, J.-M., Tzvetkov, N.: On global Strichartz estimates for non trapping metrics. *J. Funct. Analysis.* **254**, 1661–1682 (2008)
- [4] Burq, N., Gérard, P., Tzvetkov, N.: Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds. *Amer. J. Math.* **126**, 569–605 (2004)

- [5] Burq, N., Guillarmou, C., Hassell, A.: Strichartz estimates without loss on manifolds with hyperbolic trapped geodesics. *Geom. Funct. Anal.* **20**, 627–656 (2010)
- [6] Cazenave, T.: Semilinear Schrödinger equations. Courant. Lect. Notes Math. vol. 10, AMS, Providence, RI, 2003.
- [7] Christ, M., Kiselev, A.: Maximal functions associated to filtrations. *J. Funct. Analysis* **179**, 409–425 (2001)
- [8] D’Ancona, P., Fanelli, L., Vega, L., Visciglia, N.: Endpoint Strichartz estimates for the magnetic Schrödinger equation. *J. Funct. Analysis* **258**, 3227–3240 (2010)
- [9] Doi, S.: Smoothness of solutions for Schrödinger equations with unbounded potentials. *Publ. Res. Inst. Math. Sci.* **41**, 175–221 (2005)
- [10] Erdoğan, M., Goldberg, M., Schlag, W.: Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions. *Forum Math.* **21**, 687–722 (2009)
- [11] Fujiwara, D.: Remarks on convergence of the Feynman path integrals. *Duke Math. J.* **47**, 559–600 (1980)
- [12] Ginibre, J., Velo, G.: The global Cauchy problem for the non linear Schrödinger equation. *Ann. IHP-Analyse non linéaire.* **2**, 309–327 (1985)
- [13] Hassell, A., Tao, T., Wunsch, J.: Sharp Strichartz estimates on non-trapping asymptotically conic manifolds. *Amer. J. Math.* **128**, 963–1024 (2006)
- [14] Isozaki, H., Kitada, H.: Modified wave operators with time independent modifiers. *J. Fac. Sci. Univ. Tokyo.* **32**, 77–104 (1985)
- [15] Keel, M., Tao, T.: Endpoint Strichartz Estimates. *Amer. J. Math.* **120**, 955–980 (1998)
- [16] Marzuola, J., Metcalfe, J., Tataru, D.: Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations. *J. Funct. Analysis* **255**, 1497–1553 (2008)
- [17] Mizutani, H.: Strichartz estimates for Schrödinger equations on scattering manifolds. To appear in *Comm. Partial Differential Equations*.
- [18] Robbiano, L., Zuily, C.: Strichartz estimates for Schrödinger equations with variable coefficients. *Mém. SMF. Math. Fr. (N.S.), No. 101–102*, 1–208 (2005)
- [19] Robert, D.: *Autour de l’approximation semi-classique*. *Progr. Math.* **68** Birkhäuser, Basel, 1987
- [20] Robert, D.: Relative time delay for perturbations of elliptic operators and semiclassical asymptotics. *J. Funct. Analysis* **126**, 36–82 (1994)
- [21] Smith, H., Sogge, C.D.: Global Strichartz estimates for nontrapping perturbations of the Laplacian. *Comm. Partial Differential Equations* **25**, 2171–2183 (2000)
- [22] Staffilani, G., Tataru, D.: Strichartz estimates for a Schrödinger operator with non smooth coefficients. *Comm. Partial Differential Equations* **27**, 1337–1372 (2002)
- [23] Strichartz, R.: Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations. *Duke Math. J.* **44**, 705–714 (1977)

- [24] Tataru, D.: Parametrixes and dispersive estimates for Schrödinger operators with variable coefficients. *Amer. J. Math.* **130**, 571–634 (2008)
- [25] Yajima, K.: Existence of solutions for Schrödinger evolution equations. *Comm Math. Phys.* **110**, 415–426 (1987)
- [26] Yajima, K.: Schrödinger evolution equation with magnetic fields. *J. d'Anal. Math.* **56**, 29–76 (1991)
- [27] Yajima, K.: Boundedness and continuity of the fundamental solution of the time dependent Schrödinger equation with singular potentials. *Tohoku Math. J.* **50**, 577–595 (1998)
- [28] Yajima, K.: On the behavior at infinity of the fundamental solution of time dependent Schrödinger equation. *Rev. Math. Phys.* **13**, 891–920 (2001)
- [29] Yajima, K., Zhang, G.: Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity. *J. Differential Equations.* **202**, 81–110 (2004)